

Identifying codes in line graphs¹

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Identifying codes

Let $N[u]$ be the closed neighbourhood of vertex u .

Identifying code of G (Karpovsky, Chakrabarty, Levitin, 1998)

Subset C of $V(G)$ such that :

- C is a **dominating set** in G : $\forall u \in V(G), N[u] \cap C \neq \emptyset$, and
- C is a **separating code** in G : $\forall u \neq v$ of $V(G)$,
 $N[u] \cap C \neq N[v] \cap C$

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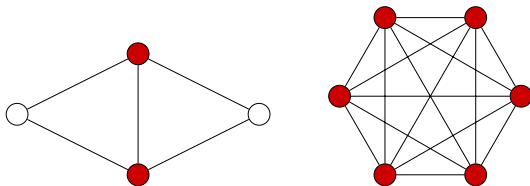
Identifying code number $\gamma^{\text{ID}}(G)$: minimum cardinality of an identifying code of G

Identifying codes

Not all graphs have an identifying code !

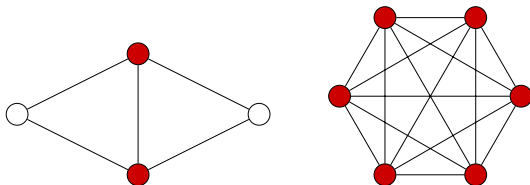
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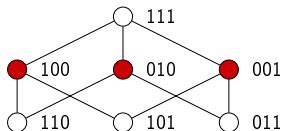
Twins = pair u, v such that $N[u] = N[v]$.

A graph is **identifiable** iff it is **twin-free** (i.e. it has no twins)

Bounds

Theorem (Karpovsky, Chakrabarty, Levitin, 1998)

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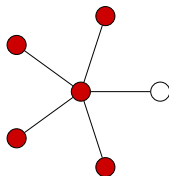
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Edge-identifying codes, definition

Let $N[e]$ be the set of edges f s.t. $e = f$ or e, f are incident to a common vertex

Definition

Edge-identifying code of G (without isolated vertices). Subset C_E of $E(G)$ such that :

- C_E is an **edge-dominating set** in G : $\forall e \in E(G)$, $N[e] \cap C_E \neq \emptyset$, and
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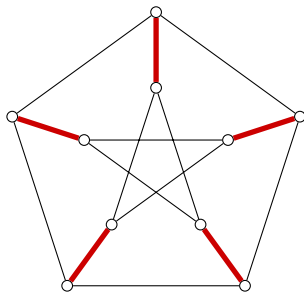
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Edge-identifying code number $\gamma^{\text{EID}}(G)$: minimum cardinality of an edge-identifying code of G

Edge-identifying code - example



Line graph

Definition

Line graph of G : Edge-adjacency graph of G . Denoted $\mathcal{L}(G)$

$$V(\mathcal{L}(G)) = E(G)$$

$e \sim e'$ in $\mathcal{L}(G)$ iff e and e' are incident to a common vertex in G

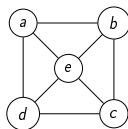
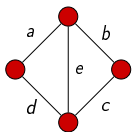
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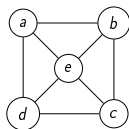
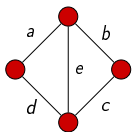
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Remark

Edge-identifying code of $G \iff$ Identifying code of $\mathcal{L}(G)$

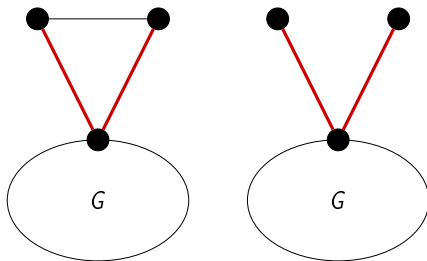
$$\gamma^{\text{EID}}(G) = \gamma^{\text{ID}}(\mathcal{L}(G))$$

Edge-identifiable graphs

Remark

Not all graphs have an edge-identifying code! *Pendant* = pair of twin edges.

A graph is **edge-identifiable** iff it is **pendant-free** (and simple).



Lower bounds

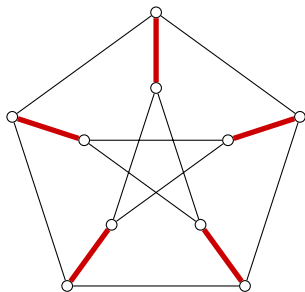
Theorem

Let G be an edge-identifiable graph. Then $\gamma^{\text{EID}}(G) \geq \frac{|V(G)|}{2}$.
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Theorem

Let G be an edge-identifiable graph with an edge-identifying code of

$$\text{size } k, \text{ then } |E(G)| \leq \begin{cases} \binom{\frac{4}{3}k}{2}, & \text{if } k \equiv 0 \pmod{3} \\ \binom{\frac{4}{3}(k-1)+1}{2} + 1, & \text{if } k \equiv 1 \pmod{3} \\ \binom{\frac{4}{3}(k-2)+2}{2} + 2, & \text{if } k \equiv 2 \pmod{3} \end{cases}$$

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Main idea of the proof : we show that in the extremal case $G[C_E]$ has to be a disjoint union of P_4 's + some leftovers.

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Lower bound - question

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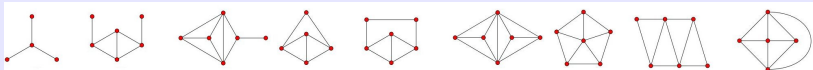
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$$\gamma^{ID}(\mathcal{L}(G)) > \frac{3\sqrt{2}}{4} \sqrt{|V(\mathcal{L}(G))|}$$

Theorem (Beineke, 1970)

G is a line graph **iff** it has none of the following graphs as induced subgraph :



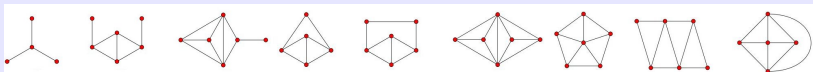
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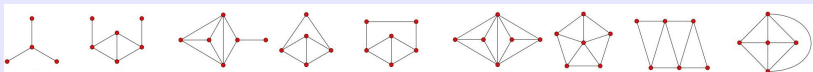
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Does it hold for an other class containing line graphs?

An upper bound

Theorem

If G edge-identifiable, $\gamma^{\text{EID}}(G) \leq 2|V(G)| - 3$.

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This is almost tight since $\gamma^{\text{EID}}(K_{2,n}) = 2n - 2 = 2|V(K_{2,n})| - 6$.

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Corollary

If G is an edge-identifiable graph **with average degree** $\bar{d}(G) \geq 5$, then $\gamma^{\text{ID}}(\mathcal{L}(G)) \leq n - \frac{n}{\Delta(\mathcal{L}(G))}$ where $n = |V(\mathcal{L}(G))|$.

Conjecture (Foucaud, Klasing, Kosowski, Raspaud, 2009)

Let G be a connected identifiable graph on n vertices and of maximum degree Δ . Then $\gamma^{\text{ID}}(G) \leq n - \frac{n}{\Delta} + O(1)$.

Complexity

EDGE-IDCODE

INSTANCE : A graph G and an integer k .

QUESTION : Does G have an edge identifying code of size at most k ?

IDCODE

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Complexity

Theorem

EDGE-IDCODE is NP-complete, even for planar subcubic bipartite graphs of arbitrarily large girth.

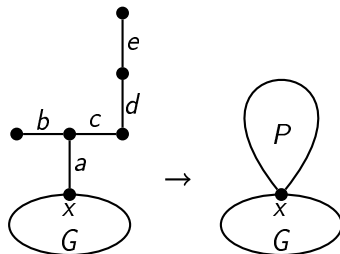
Proof by reduction from

PLANAR ($\leq 3, 3$)-SAT

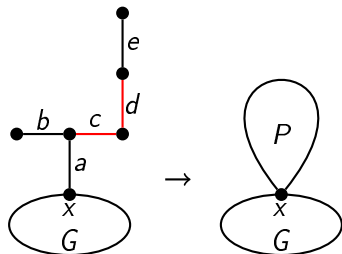
INSTANCE : A collection \mathcal{Q} of clauses over a set X of boolean variables. Two or three distinct literals per clause. Every variable appears in exactly 3 clauses and each non-negated literal appears in exactly 2 clauses.

QUESTION : Is there a truth assignment of the variables of X such that each clause contains at least one true literal?

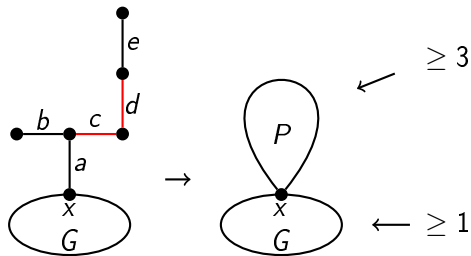
Reduction



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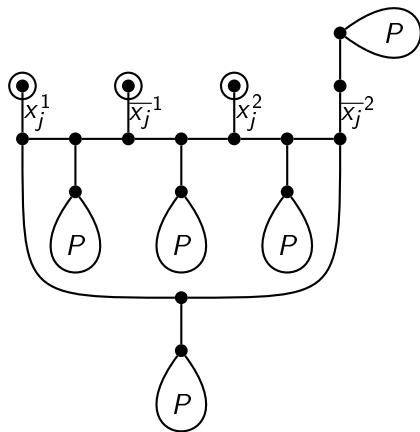
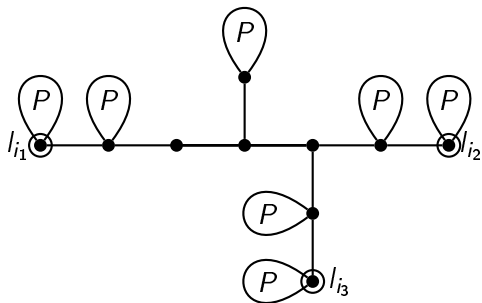


Reduction



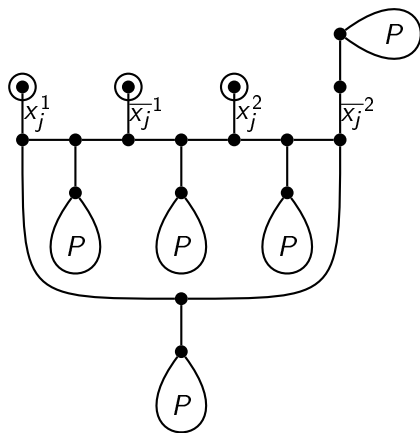
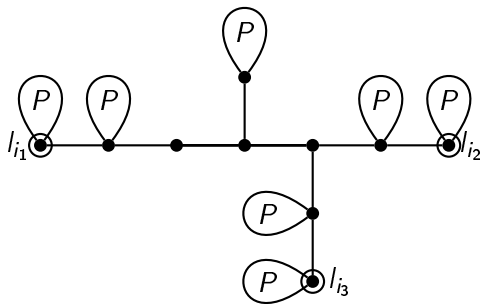
Clause gadget

Variable gadget



Clause gadget

Variable gadget



\mathcal{C} is satisfiable if and only if G contains an edge identifying code \mathcal{I} of size $k = 25|\mathcal{C}| + 22|X|$.

Complexity

Corollary

IDCODE is NP-complete even when restricted to perfect 3-colorable planar line graphs of maximum degree 4.