

Watching Systems in Graphs: an Extension of Identifying Codes

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- (7. Watching systems in the king grid = next talk, by David Auger)

[1] AUGER, CHARON, HUDRY, LOBSTEIN:
Maximum size of a minimum watching system
and the graphs achieving the bound,
Rapport interne Telecom ParisTech-2010D011,
Paris, France, 40 pages, March 2010.

Also: *Discrete Applied Mathematics*, to appear.

[2] AUGER, CHARON, HUDRY, LOBSTEIN:
Watching systems in graphs: an extension of
identifying codes,

Discrete Applied Mathematics, to appear.

<http://hal.archives-ouvertes.fr/hal-00481469>

[3] [http://perso.telecom-paristech.fr/~lobstein/
debutBIBidetlocdom.pdf](http://perso.telecom-paristech.fr/~lobstein/debutBIBidetlocdom.pdf)

PLEASE SEND ME MORE REFERENCES!!

1. Introduction and definitions

Watching systems *in the case* $r = 1$ (easy generalization) :

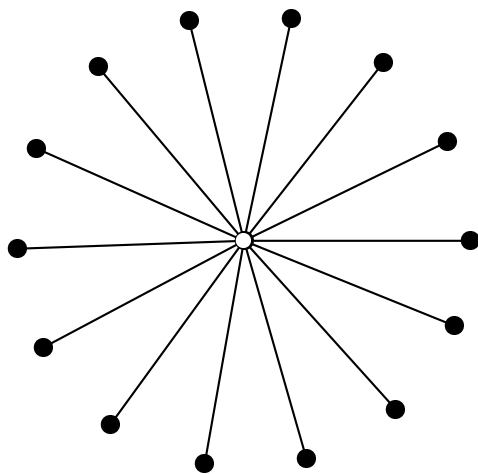
instead of checking all its closed neighbourhood (= ball of radius 1), the codeword (now called *watcher*) checks a *subset* of its neighbourhood (= *watching zone* or *checking zone*).

It is a **couple** $w = (x \in V(G), Z \subseteq B_1(x))$.

NOTE THAT:

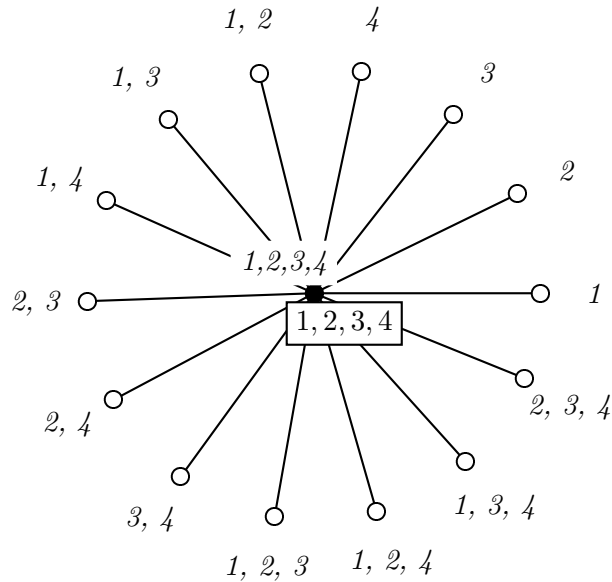
- watching systems always exist
- when an identifying code also exists, its size is at least that of a minimum watching system
- several watchers can be located on the same vertex (with different watching zones)
- a watching system remains so **if we add edges**

EXAMPLE:



Star with **15** vertices, minimum identifying code of size **14**

One can do with 4 watchers !



Star with **15** vertices, minimum watching system of size **4**

2. First properties of watching systems

Let respectively $w(G)$, $\gamma(G)$ and $i(G)$ denote the minimum sizes of a watching system, of a dominating set and, when it exists, of an identifying code in G .

These parameters will be called *watching number*, *domination number*, and *identifying number*, respectively.

Theorem 1. *For any graph G , we have:*

$$\lceil \log_2(|V(G)| + 1) \rceil \leq w(G).$$

For any twin-free graph G , we have (already mentioned):

$$w(G) \leq i(G).$$

Theorem 2. *For any graph G , we have:*

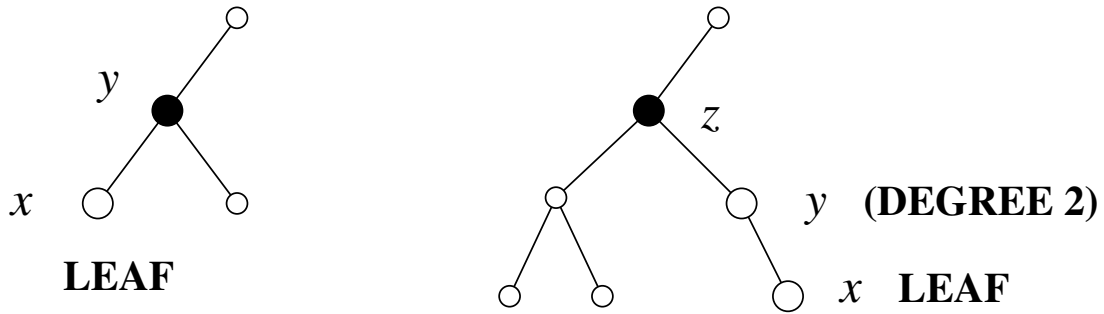
$$\gamma(G) \leq w(G) \leq \gamma(G) \cdot \lceil \log_2(\Delta(G) + 2) \rceil,$$

where $\Delta(G)$ denotes the maximum degree of G .

Proof. If \mathcal{W} is a watching system, then the set of the watchers' locations in \mathcal{W} is a dominating set, so we have the left-hand inequality. On the other hand, if we have a dominating set $\Gamma \subseteq V(G)$ of size $\gamma(G)$, we can identify all vertices simply by locating enough watchers at every vertex of Γ . One just has to notice that in order to identify a vertex v and its (at most) $\Delta(G)$ neighbours, we need at most $p := \lceil \log_2(\Delta(G) + 2) \rceil$ watchers, since a set with p elements has at least $\Delta(G) + 1$ nonempty subsets. \square

3. An upper bound for the watching number

It is known that $i(G) \leq |V(G)| - 1$ for any connected twin-free graph with at least three vertices, and that this bound is reached, for instance, by the star. We prove that a much smaller number of watchers is needed, namely $2n/3$; we will use the following three lemmata.



Lemma 3. (already mentioned) *Let G be a graph and H be a partial graph of G , i.e., with $V(H) = V(G)$ and $E(H) \subseteq E(G)$. Then*

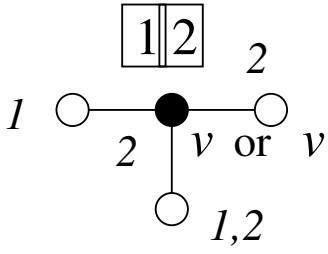
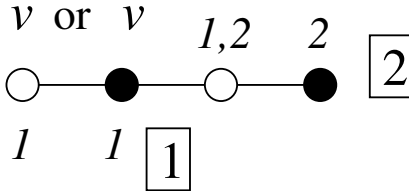
$$w(H) \geq w(G).$$

Note that this monotony property does **not** hold in general for identifying codes.

Lemma 4. *Let T be a tree, x be a leaf of T , and y be the neighbour of x .*

(a) *There exists a minimum watching system for T with one watcher located at y .*

(b) *If y has degree 2, there exists a minimum watching system for T with one watcher located at z , the second neighbour of y .*



Lemma 5. *Let T be a tree with four vertices, and let v be a vertex of T ; there exists a set \mathcal{W} of two watchers such that*

- *the vertices in $V(T) \setminus \{v\}$ are checked and pairwise separated by \mathcal{W} — in this case, we shall say, with a slight abuse of notation, that \mathcal{W} is a watching system of $V(T) \setminus \{v\}$;*
- *the vertex v is checked by at least one watcher.*

Theorem 6. *If G is a connected graph of order n :*

- *If $n = 1$, $w(G) = 1$.*
- *If $n = 2$ or $n = 3$, $w(G) = 2$.*
- *If $n = 4$ or $n = 5$, $w(G) = 3$.*
- *If $n \notin \{1, 2, 4\}$, $w(G) \leq \frac{2n}{3}$.*

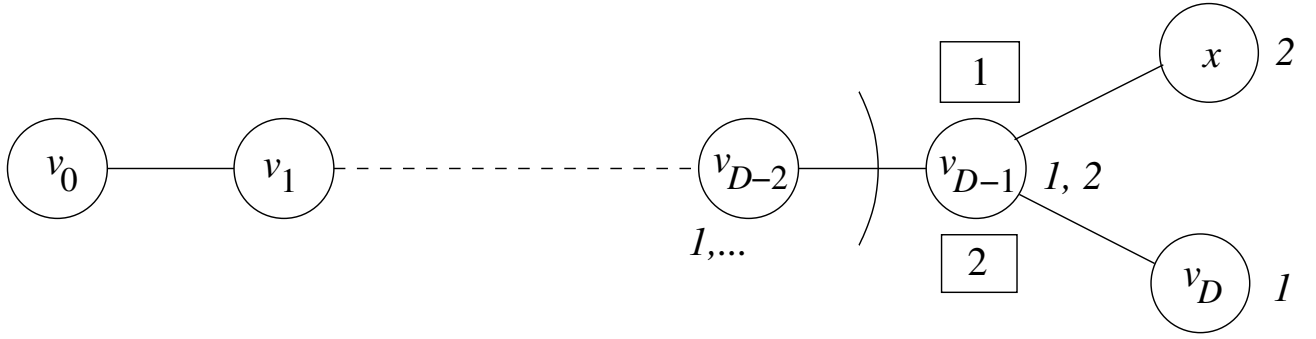
Proof. Small cases (up to 5) are easy.

We proceed **by induction** on n .

We assume that $n \geq 6$ and that the theorem is true for any connected graph of order less than n .

Let G be a connected graph of order n . Let T be a spanning tree of G ; we will prove that $w(T) \leq \frac{2n}{3}$ and then the theorem will result from Lemma 3. We denote by D the diameter of T and we consider a path $v_0, v_1, v_2, \dots, v_{D-1}, v_D$ of T , with length D .

We distinguish between four cases, according to some conditions on the degrees of v_{D-1} and v_{D-2} .



- *First case: the degree of v_{D-1} is equal to 3*

The vertex v_{D-1} is adjacent to a vertex x other than v_{D-2} and v_D ; because D is the diameter, clearly x and v_D are leaves of T (see **Figure**). We consider the tree obtained by removing x , v_{D-1} and v_D from T ; this new tree T' has order $n - 3$.

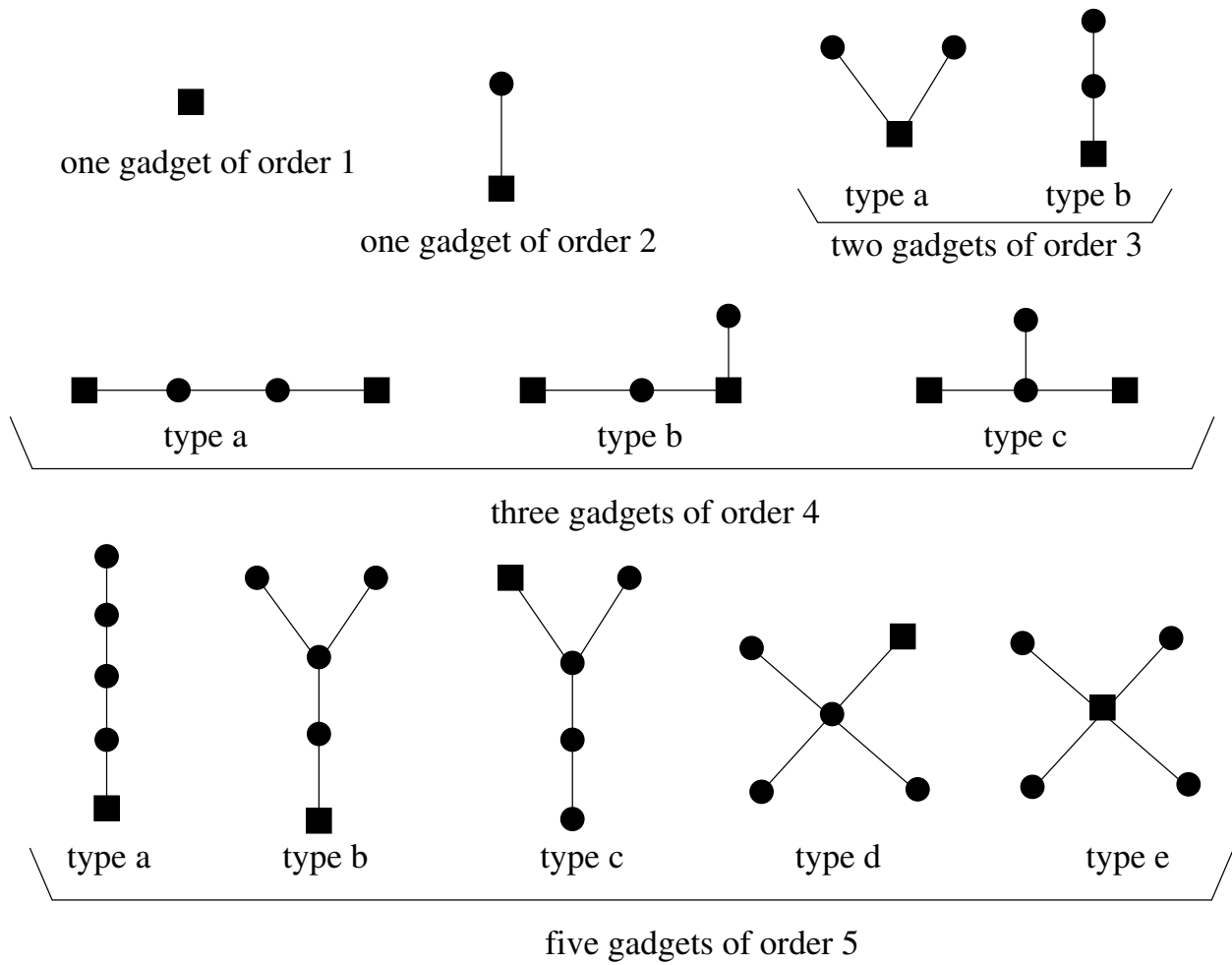
If $n \geq 8$ or if $n = 6$, we consider a minimum watching system \mathcal{W} for T' ; if $n = 7$, then T' is of order 4, and, using Lemma 5, we choose a set \mathcal{W} of two watchers which is a watching system for $V(T') \setminus \{v_{D-2}\}$ and checks the vertex v_{D-2} .

Then for T , in both cases, we add to \mathcal{W} two watchers $w_1 = (v_{D-1}, \{v_{D-2}, v_{D-1}, v_D\})$ and $w_2 = (v_{D-1}, \{v_{D-1}, x\})$. On the **Figure**, we rename 1 and 2 these watchers. Then $\mathcal{W} \cup \{w_1, w_2\}$ is a watching system for T . So, $w(T) \leq |\mathcal{W}| + 2 \leq w(T') + 2$.

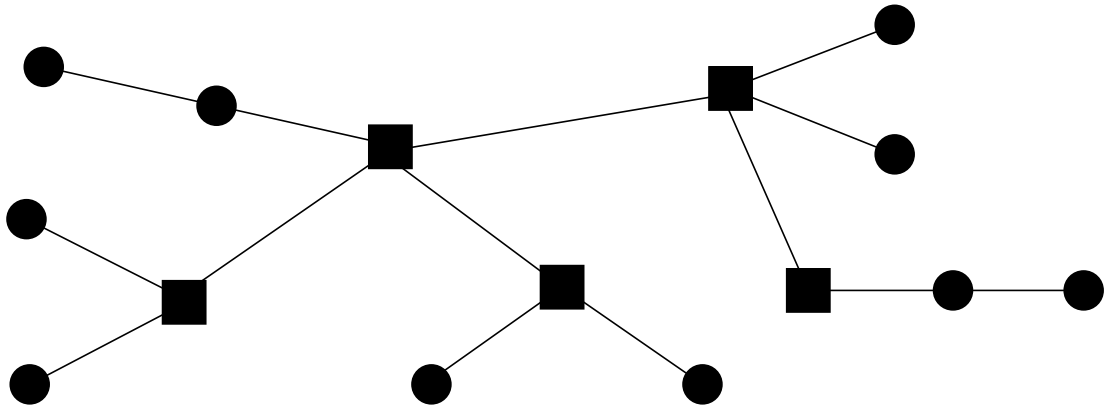
Now we use the induction hypothesis:

if $n \geq 8$ or $n = 6$, then $w(T) \leq \frac{2}{3}(n - 3) + 2 = \frac{2n}{3}$;

and if $n = 7$, then $w(T) \leq 2 + 2 = 4 < \frac{2}{3} \times 7$. □



We can characterize the *trees* T
with n vertices and $w(T) = \lfloor \frac{2n}{3} \rfloor$.

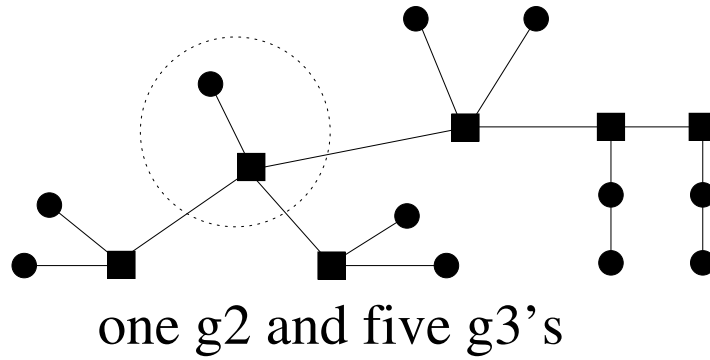
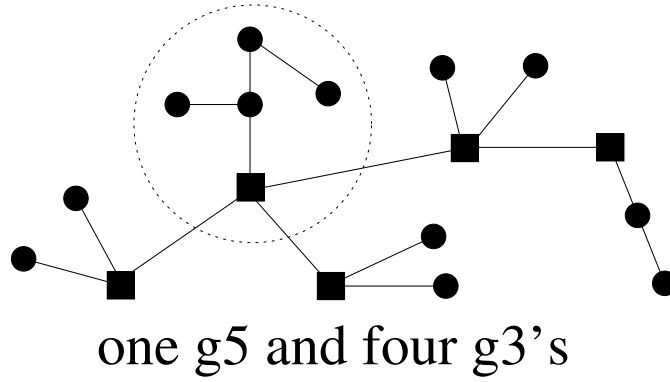


Tree T with 15 vertices and $w(T) = 10$

Theorem 7. *Let T be a tree of order $n = 3k$ for $k \geq 1$.*

We have:

$w(T) = 2k \Leftrightarrow T$ can be obtained by choosing k gadgets of order 3 and joining these gadgets by their binding vertices to obtain a tree.



Trees T with 17 vertices and $w(T) = 11$

Theorem 8. *Let T be a tree of order $n = 3k + 2$ for $k \geq 1$.*

We have:

$w(T) = 2k + 1 \Leftrightarrow T$ can be obtained by choosing one gadget of order 2 and k gadgets of order 3, or one gadget of order 5 and $k - 1$ gadgets of order 3, and joining these gadgets by their binding vertices to obtain a tree.

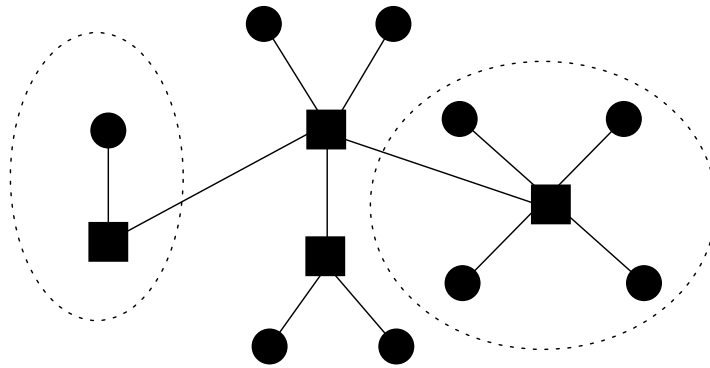
Theorem 9. *Let T be a tree of order $n = 3k + 1$ for $k \geq 2$.*

We have:

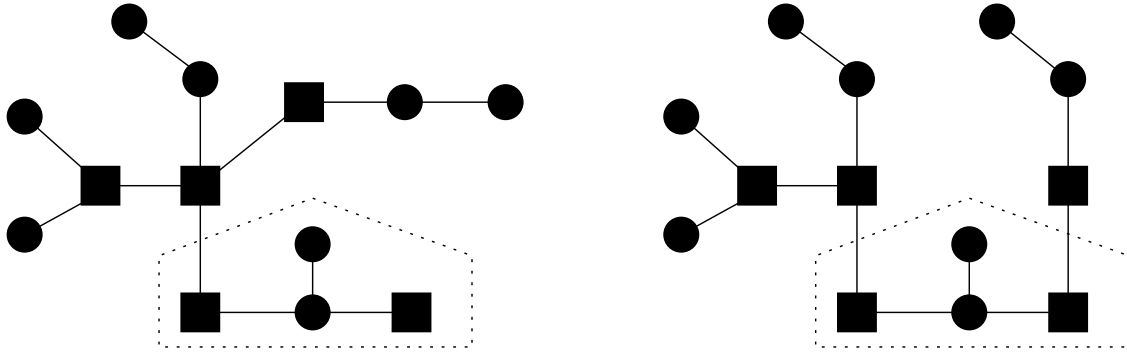
$w(T) = 2k \Leftrightarrow T$ can be obtained by choosing

- (i) *two gadgets of order 2 and $k - 1$ gadgets of order 3,*
- (ii) *or one gadget of order 2, one gadget of order 5 and $k - 2$ gadgets of order 3,*
- (iii) *or two gadgets of order 5 and $k - 3$ gadgets of order 3,*
- (iv) *or one gadget of order 1 and k gadgets of order 3,*
- (v) *or one gadget of order 4 and $k - 1$ gadgets of order 3,*

and joining these gadgets by their binding vertices to obtain a tree.



two g_3 's, one g_2 and one g_5



three g_3 's and one g_4

Trees T with 13 vertices and $w(T) = 8$

We can almost characterize the graphs for which the bound is tight: we can do it in the cases $n = 3k$, $k \geq 1$, and $n = 3k + 2$, $k \geq 1$; the case $n = 3k + 1$ is more complex, and we are only able to state a conjecture for $k \geq 6$.

Definition 10. *A connected graph G is said to be maximal if, when we add any edge to G , we obtain a graph G' verifying: $w(G') < w(G)$.*

We denote by $\omega(n)$ the maximum minimum number of watchers needed in a connected graph of order n , i.e.,

$$\omega(n) = \max\{w(G) : G \text{ connected of order } n\}.$$

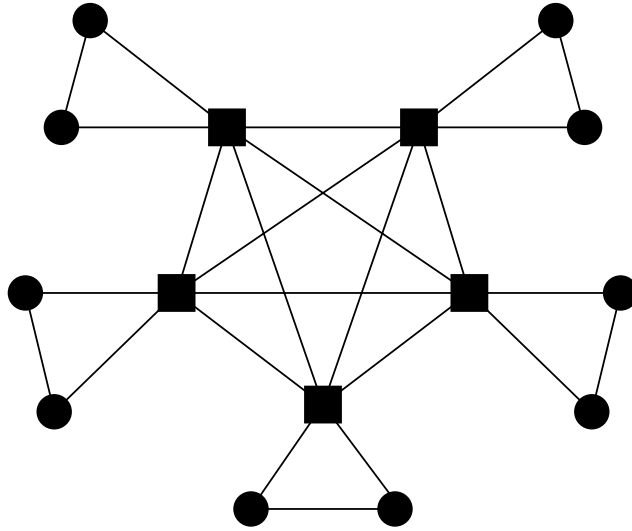
We have just established that

$$\omega(n) = \lfloor \frac{2n}{3} \rfloor \text{ for } n \notin \{1, 2, 4\},$$

and we have characterized the trees of order n reaching $\omega(n)$.

Now, we want to describe all the maximal connected graphs of order n which reach $\omega(n)$.

Using Lemma 3, the graphs of order n which reach $\omega(n)$ are exactly the connected partial graphs of the maximal connected graphs of order n reaching $\omega(n)$.



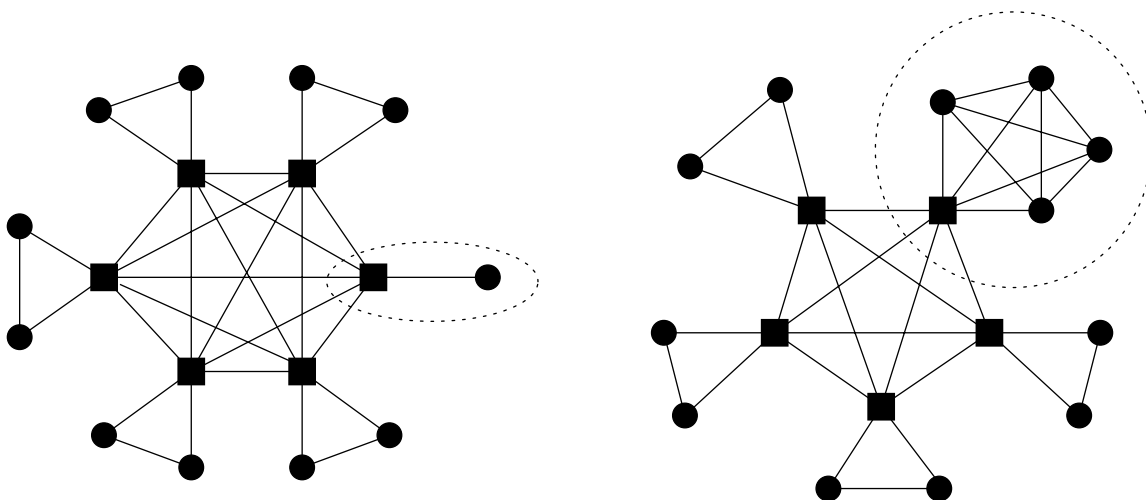
The unique maximal graph of order 15 reaching the bound $\omega(15) = 10$.

Theorem 11. *Let k be an integer, $k \geq 1$, and G be a maximal graph of order $3k$.*

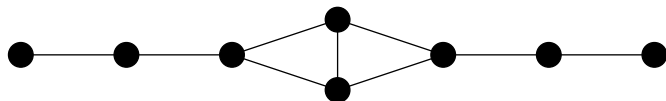
We have:

$w(G) = 2k \Leftrightarrow G$ is obtained by taking a collection of k K_3 's, choosing one vertex named a binding vertex in each K_3 , and connecting these k binding vertices by K_k .

The two maximal graphs of order 17 reaching the bound



The additional maximal graph of order 8 reaching the bound

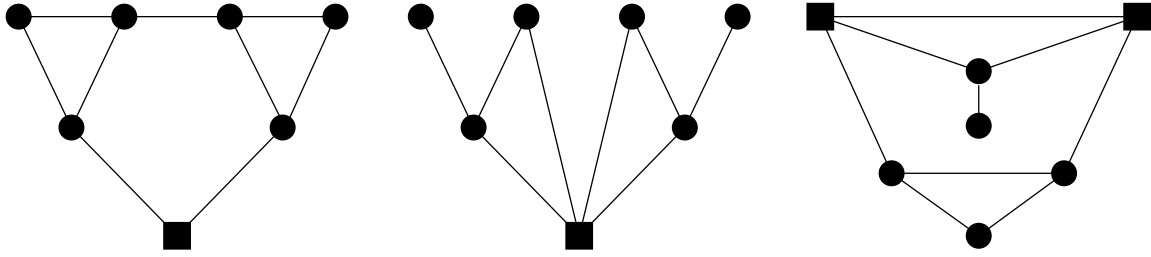


Theorem 12. (a) Let k be an integer, $k \geq 3$, and G be a maximal graph of order $3k + 2$. We have:

$w(G) = 2k + 1 \Leftrightarrow G$ is obtained by taking a collection of k K_3 's and one K_2 , or $k - 1$ K_3 's and one K_5 , choosing one vertex named a binding vertex in each of these complete graphs, and connecting these binding vertices by K_{k+1} if we have taken a K_2 , and by K_k if we have taken a K_5 .

(b) If G is a maximal graph of order 8, then we have:
 $w(G) = 5 \Leftrightarrow G$ is the graph given above, or G is obtained by following the rules given in Case (a), for $k = 2$.

(c) The only maximal graph G of order 5 with $w(G) = 3$ is the clique K_5 .



New gadgets (of order 7)

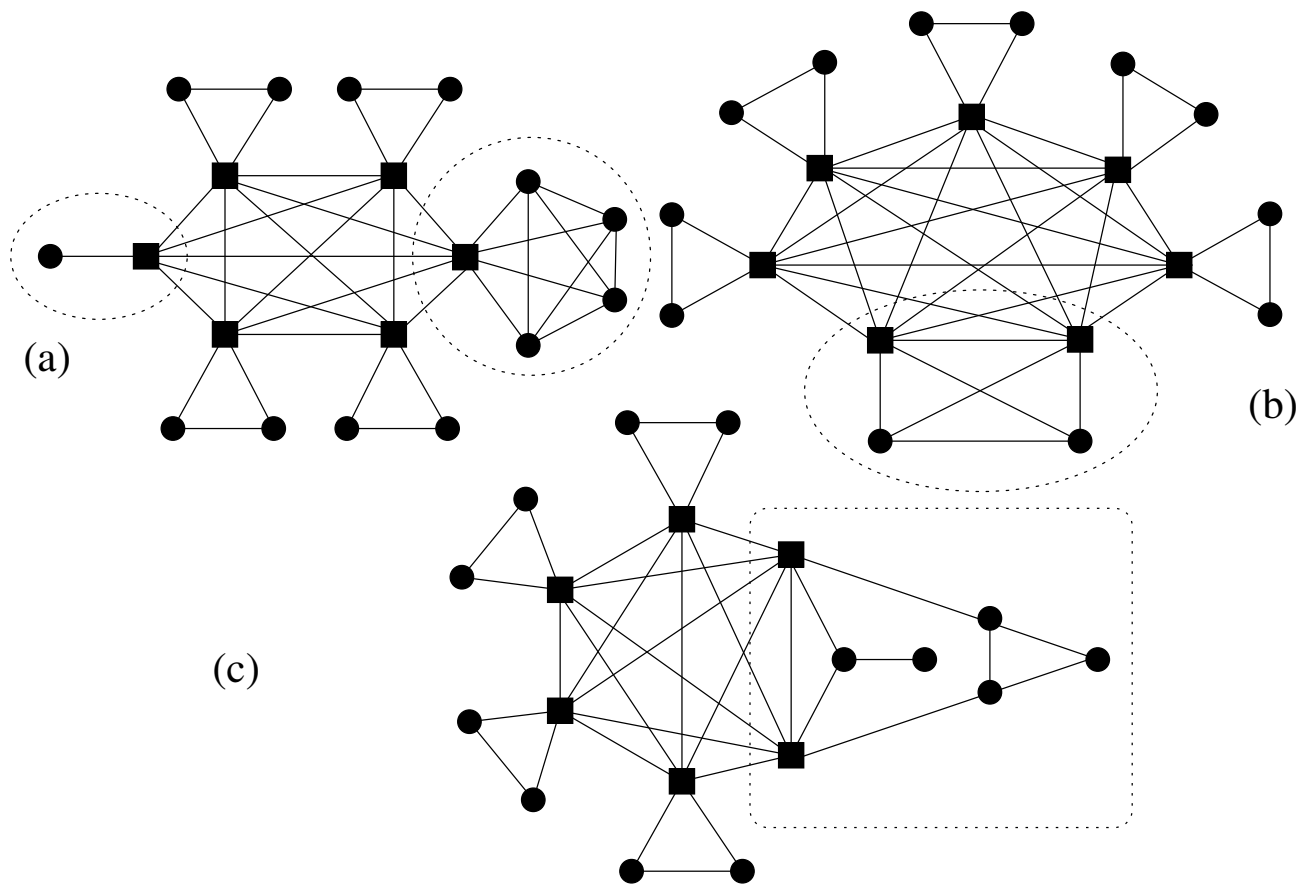
Conjecture 13. *Let k be an integer, $k \geq 6$, and G be a maximal graph of order $3k + 1$.*

We have:

$w(G) = 2k \Leftrightarrow G$ is obtained by:

- (i) taking two K_2 's and $k - 1$ K_3 's,
- (ii) or taking one K_2 , one K_5 and $k - 2$ K_3 's,
- (iii) or taking two K_5 's and $k - 3$ K_3 's,
- (iv) or taking one K_4 and $k - 1$ K_3 's,
- (v) or taking one $g\gamma$ and $k - 2$ K_3 's,

choosing one vertex named a binding vertex on each of the complete components K_i , except on K_4 for which we choose two binding vertices, taking for the $g\gamma$ one or two binding vertex(ices) according to its type, and connecting these binding vertices to form a complete graph with them.

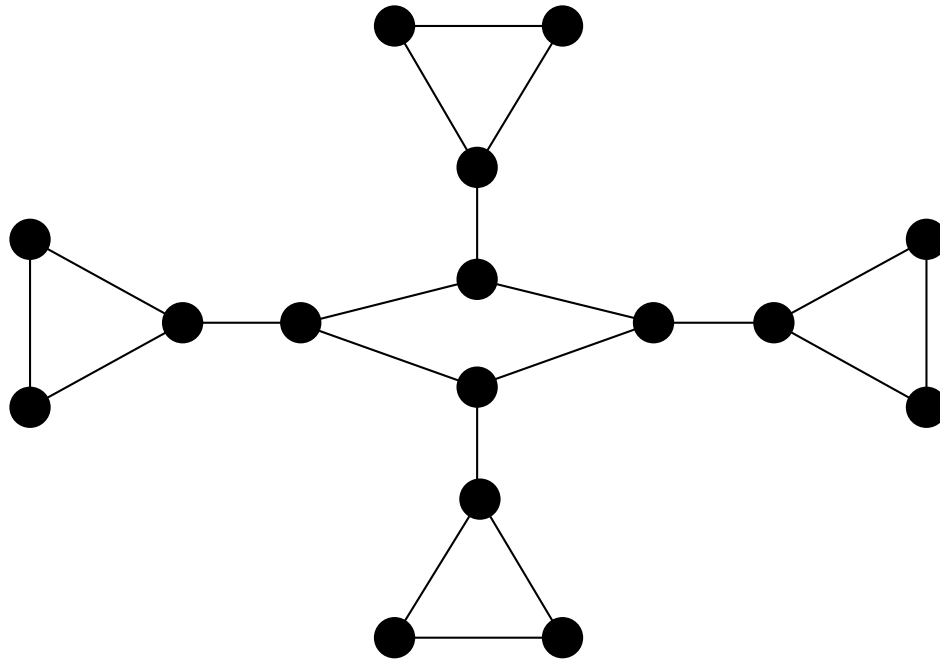


The graphs above are graphs of order 19 reaching the bound $\omega(19) = 12$:

(a) with one K_2 , one K_5 and four K_3 's;

(b) with one K_4 and five K_3 's;

(c) with one g_7 and four K_3 's;



For $n = 3k+1$ with $k \leq 5$, there are maximal graphs needing $2k$ watchers which are not of the form described in the conjecture.

Above is a certified example for $n = 16$.

4. Complexity: NP-completeness

Vertex cover in a graph G : \mathcal{C} is a vertex cover if

$$\forall e = xy \in E(G), \quad x \in \mathcal{C} \text{ or } y \in \mathcal{C}.$$

It is well known that the problem of finding the minimum cardinality of a vertex cover in a given graph is *NP*-hard, even when restricted to the class of **planar** graphs whose **maximum degree is at most 3**, class which we denote by **PL(3)**.

With some additional work, we can also get rid of the vertices with degree one, and restrict ourselves to the class **PL'(3)** of all **planar** graphs where every vertex **has degree 2 or 3**.

Then the following decision problem is *NP*-complete:

MIN VERTEX COVER IN PL'(3)

- **INSTANCE:** A graph $G \in \text{PL}'(3)$ and an integer k ;
- **QUESTION:** Is there a vertex cover for G with size at most k ?

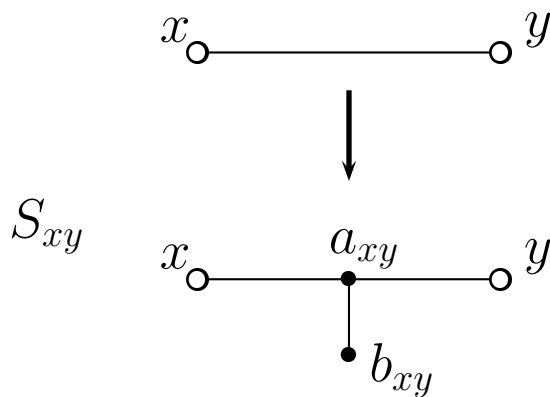
MIN WATCHING SYSTEM IN PL(3)

- **INSTANCE:** A planar graph G' , with maximum degree at most 3, and an integer k' ;
- **QUESTION:** Is there a watching system for G' with size at most k' ?

Theorem 14. MIN WATCHING SYSTEM IN PL(3) is NP-complete.

Proof. 1. MIN WATCHING SYSTEM IN PL(3) \in NP.

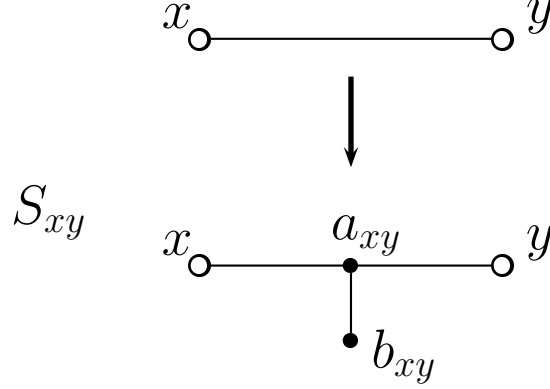
2. Take G (with m edges) and k , instance of MIN VERTEX COVER IN PL'(3). Construct G' by **replacing every edge xy** of G by **the structure S_{xy}** depicted below.



G' has $n + 2m$ vertices and $3m$ edges: the construction is **polynomial**. Moreover, if $G \in \text{PL}'(3)$, then $G' \in \text{PL}(3)$.

We set $k' = k + m$. The proof is complete if we prove that

G admits a vertex cover of size at most k if and only if G' admits a watching system of size at most k' .



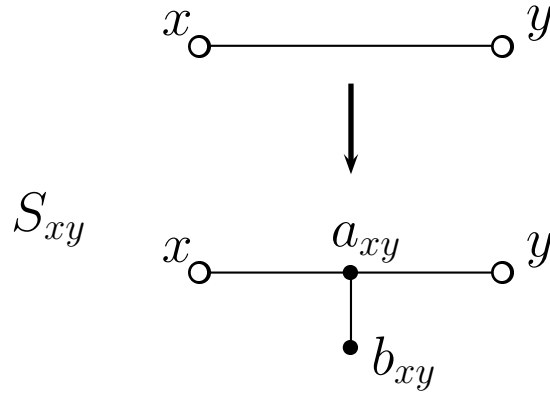
Assume first that \mathcal{C} is a vertex cover of G .

We define a watching system \mathcal{W} in G' as follows:

- for every vertex x of $V(G)$ such that $x \in \mathcal{C}$, we add the watcher $(x, N_{G'}[x])$ to \mathcal{W} ;
- for every edge $e = xy$ of G , we add the watcher $(a_{xy}, N_{G'}[a_{xy}])$ to \mathcal{W} .

It is easy to see that \mathcal{W} is a watching system in G' . Consider a vertex x in G ; since it has degree at least 2 in G , it is adjacent to at least two vertices y_1 and y_2 in G . So the corresponding vertex x in G' is checked by, at least, the two watchers located at a_{xy_1} and a_{xy_2} , belonging respectively to the structures S_{xy_1} and S_{xy_2} , and thus x is identified by \mathcal{W} . Also note that for every edge $e = xy$ of G , since either x or y belong to the vertex cover \mathcal{C} , there is a watcher in \mathcal{W} that separates a_{xy} from b_{xy} .

Thus G' admits a watching system with size $|\mathcal{C}| + m \leq k'$.



Let $V'_{xy} = \{a_{xy}, b_{xy}\}$.

Conversely, assume that \mathcal{W} is a watching system of G' of size at most k' .

Consider an edge $xy \in E(G)$ and the watchers located in the structure S_{xy} of G' . Then:

- if no watcher is located at x nor y , there must be at least two watchers located in V'_{xy} ;
- if at least one watcher is located at x or y , we still need at least one watcher in V'_{xy} .

So if we denote by \mathcal{C} the set of vertices $x \in V(G)$ such that \mathcal{W} contains a watcher located at x , and by p the number of edges xy of G with $x \notin \mathcal{C}$ and $y \notin \mathcal{C}$, we have

$$|\mathcal{C}| \leq |\mathcal{W}| - 2p - (m - p) \leq k' - m - p \leq k - p.$$

Therefore if we add to \mathcal{C} one vertex for every uncovered edge of G , we get a vertex cover of G of size at most k . \square

5. Watching systems in paths and cycles

Theorem 15. *For all $n \geq 1$, we have:*

$$w(P_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Constructions proving that $\lceil \frac{n+1}{2} \rceil$ is an upper bound are easy to find; actually it is sufficient to use identifying codes: on paths, watching systems are **no better** than identifying codes, except for $n = 2$, when no identifying code exists.

Theorem 16. *We have $w(C_4) = 3$, and for $n = 3$ and all $n \geq 5$:*

$$w(C_n) = \left\lceil \frac{n}{2} \right\rceil.$$

For identifying codes, we know that:

the cycle of length three admits no identifying code,

$$i(C_4) = i(C_5) = 3,$$

$$i(C_n) = \frac{n}{2} \text{ when } n \text{ is even, } n \geq 6,$$

$$i(C_n) = \frac{n+3}{2} \text{ when } n \text{ is odd, } n \geq 7.$$

So $i(C_n) = w(C_n)$ when $n = 5$ or n is even, $n \geq 4$,
and $i(C_n) = w(C_n) + 1$ when n is odd, $n \geq 7$.

Conclusion: Watching systems do not bring much !

6. Watching sets of vertices

Generalization to $r \geq 1$ and $\ell \geq 1$:

$(r, \leq \ell)$ -watching systems

Let \mathcal{W} be a set of r -watchers in G .

If $A \subset V(G)$, we define the \mathcal{W} -label of A as

$$L_{\mathcal{W}}(A) = \bigcup_{v \in A} \{\text{watchers checking } v\}.$$

$(r, \leq \ell)$ -*watching system*: all the labels of the subsets A of $V(G)$ with $1 \leq |A| \leq \ell$ are $\neq \emptyset$ and distinct.

A $(r, \leq \ell)$ -watching system is a $(r', \leq \ell')$ -watching system if $\ell' \leq \ell$ and $r' \geq r$.

ℓ -superimposed family:

$\mathcal{S} = \{S_1, S_2, \dots, S_k\}$ finite family of distinct nonempty subsets of a set X , integer $\ell \geq 1$.

\mathcal{S} is a ℓ -*superimposed family* on X if, whenever we consider two distinct sets I, J included in $\{1, \dots, k\}$ with $1 \leq |I| \leq \ell$ and $1 \leq |J| \leq \ell$, we have:

$$\bigcup_{i \in I} S_i \neq \bigcup_{j \in J} S_j.$$

If \mathcal{W} is a $(r, \leq \ell)$ -watching system in a graph G , then the family of all \mathcal{W} -labels of the vertices of G is a ℓ -superimposed family on \mathcal{W} .

The family of singletons of X is always a ℓ -superimposed family of X for all $\ell \geq 1$, and so every graph G admits a $(r, \leq \ell)$ -watching system for all $r \geq 1$ and $\ell \geq 1$, consisting of every vertex $v \in V(G)$ checking itself.

Observe that if $\ell \geq 2$ and $i \neq j$, then $S_i \subseteq S_j$ **is impossible** in a ℓ -superimposed family.

From this follows that if $|L_{\mathcal{W}}(x)| = 1$ for a vertex x in the graph with watching system \mathcal{W} , then if $\ell \geq 2$ the watcher checking x must check only x :

we will call such a watcher a *hermit*.

Wlog, we can suppose that this watcher is located at x .

6.1 $(1, \leq 2)$ -watching systems in paths and cycles

Lemma 17. *For $1 \leq k \leq 4$, the only 2-superimposed family on a set with k elements with at least k subsets is the family of k singletons.*

In other words, with k watchers, $1 \leq k \leq 4$, we can produce k valid labels, which will be singletons, and not more.

Theorem 18. *For all $n \geq 1$, the minimum size of a $(1, \leq 2)$ -watching system in the path P_n is equal to*

$$\begin{cases} n & \text{if } n \leq 10, \\ \left\lceil \frac{5(n+1)}{6} \right\rceil & \text{if } n > 10. \end{cases}$$

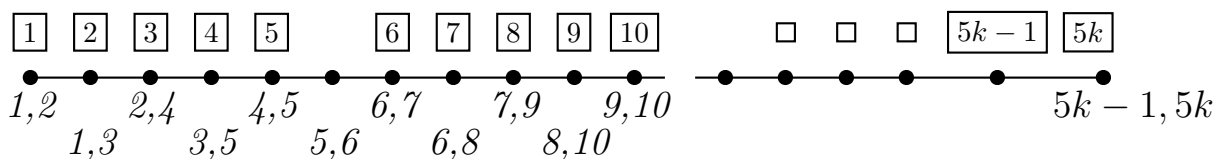
Proof. The proof for the lower bound is **by induction** on n . The study of the small cases is done using Lemma 17.

We give a **construction** that matches the lower bound. For $n \in \{1, 2, \dots, 10\}$, we need at least n watchers, and we can do it with n hermits.

For $n = 11$, see the left part of the **Figure** below.

When $n = 6k - 1$, $k \geq 3$, for which at least $5k$ watchers are necessary, use the left part of the **Figure**, add the pattern of the last six vertices to the right of the right-most vertex, x_{11} , change, for these new vertices, 6, 7, 8, 9, 10 into 11, 12, 13, 14, 15, and so on: see the right part of **Figure**.

When $n = 6k + i$, $k \geq 2$, $i \in \{0, 1, 2, 3, 4\}$, for which at least $5k + i + 1$ watchers are necessary, we use the construction for $6k - 1$ and manage with some additional work. \square



Observe also that no $(1, \leq 2)$ -identifying code (and more generally, **no $(1, \leq \ell)$ -identifying code**) exists in the path P_n , because, since $N_{P_n}[x_1] \subseteq N_{P_n}[x_2]$, the sets of vertices $\{x_2\}$ and $\{x_1, x_2\}$ cannot be separated.

Theorem 19. *For all $n \geq 3$, the minimum size of a $(1, \leq 2)$ -watching system in the cycle C_n is $\lceil \frac{5}{6}n \rceil$, except for $n = 6$, for which it is 6.*

Proof. The small cases, up to $n = 5$, are easy to handle.

The proof of the case $n = 6$ is the most cumbersome.

The lower bound in the general case makes use of the proof for the paths.

Constructions meeting the lower bound are easy: take a path with $n - 1$ vertices together with the construction of an optimal $(1, \leq 2)$ -watching system described in the proof of Theorem 18.

Add a vertex x_n which is linked to x_1 and x_{n-1} , and assign to x_n the label $\{w_1, w_2\}$, where w_1 is located at x_1 and w_2 is located at x_{n-1} (in our construction, there are always watchers located at each end of the path). You obtain a $(1, \leq 2)$ -watching system for C_n , of size $\lceil \frac{5n}{6} \rceil$, for $n \geq 7$. \square

The reason why this construction does not work for $n = 6$ is that there would be three labels, $\{1, 3\}$, $\{3, 5\}$ and $\{1, 5\}$, whose pairwise unions are equal to $\{1, 3, 5\}$.

If we compare to $(1, \leq 2)$ -identifying codes in C_n , we can see that, because $B_{C_n}(x_i, 1)$ and $B_{C_n}(x_i, 1) \cup B_{C_n}(x_{i+1}, 1)$ differ by only one vertex, x_{i+2} , this vertex, hence by symmetry all vertices, must belong to the code. Starting from $n = 7$, the **only** $(1, \leq 2)$ -identifying code in the cycle C_n is $V(C_n)$.

6.2 $(1, \leq \ell)$ -watching systems in paths and cycles for $\ell \geq 3$

Like every graph, the path P_n and the cycle C_n admit a $(1, \leq \ell)$ -watching system, which is the trivial watching system, made of all the hermits. For P_n and C_n , **this is the best we can do:**

Theorem 20. *For all $n \geq 1$ (resp., $n \geq 3$) and $\ell \geq 3$, the minimum size of a $(1, \leq \ell)$ -watching system in the path P_n (resp., the cycle C_n) is n .*

Finally, observe that for $\ell \geq 3$, **no $(1, \leq \ell)$ -identifying code exists in the cycle C_n** , because the sets of vertices $\{1, 3\}$ and $\{1, 2, 3\}$ (or more generally, $\{x, x + 2\}$ and $\{x, x + 1, x + 2\}$) cannot be separated.

		$(1, \leq 2)$	$(1, \leq \ell), \ell \geq 3$
path	id. code	does not exist	does not exist
	watching syst.	$\begin{cases} n & \text{if } n \leq 10 \\ \left\lceil \frac{5(n+1)}{6} \right\rceil & \text{if } n > 10 \end{cases}$	n
cycle	id. code	n	does not exist
	watching syst.	$\lceil \frac{5n}{6} \rceil$ if $n \neq 6$	n

Conclusion: Here, watching systems are efficient.