

# Extensional Acyclic Digraphs and Set Graphs

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# OUTLINE

SETS, DIGRAPHS, (OPEN-OUT-)SEPARATING CODES

RESULTS ON SET GRAPHS

FINDING A MINIMUM SIZE OPEN-OUT-SEPARATING CODE



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## RESULTS ON SET GRAPHS

## FINDING A MINIMUM SIZE OPEN-OUT-SEPARATING CODE



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$$\text{HF} = \bigcup_{i \in \mathbb{N}} V_i.$$



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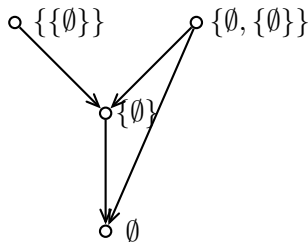


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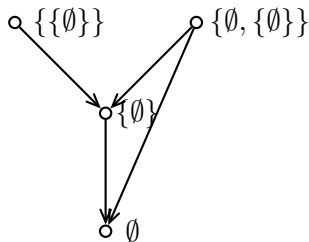
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The **transitive closure** of a set  $x$

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## DEFINITION

The **membership digraph** of the set  $x$  is

$$\left( \text{TrCl}(x), \{u \rightarrow v \mid u, v \in \text{TrCl}(x) \wedge u \ni v\} \right).$$

# SET GRAPHS

## DEFINITION

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## MAIN QUESTION

*Characterize the underlying (undirected) graphs of EADs.*

- ▶ *if a graph admits an EA orientation, we say it is a **set graph**.*



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Initial motivation: the graph-theoretic expressive power of HF:

- ▶ a set graph can be represented by the transitive closure from which it originates (e.g. in  $\{\log\}$ , CLP( $\mathcal{SET}$ ), Referee)



# SEPARATING CODES

- ▶ Given a digraph  $D$ , we say that a subset  $C \subseteq V(D)$  is a **separating code** if

$\forall u \neq v \in V(D)$ , it holds that  $N^-[u] \cap C \neq N^-[v] \cap C$ .

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- ▶ A digraph  $D$  admits an ooSC  $\Leftrightarrow D$  is extensional.



SETS, DIGRAPHS, (OPEN-OUT-)SEPARATING CODES

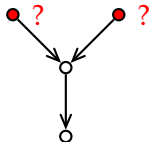
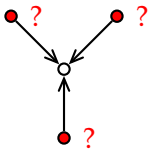
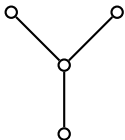
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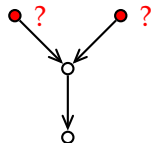
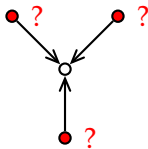
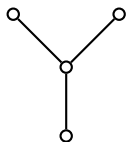
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- ▶ Every set graph is connected (since every EA orientation must have a unique sink)
- ▶ There are connected graphs which are not set graphs: e.g., the claw,  $K_{1,3}$



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### PROPOSITION

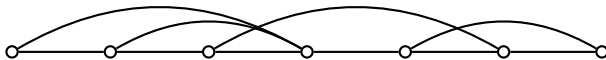
If  $G$  is a set graph, then for every  $X \subseteq V(G)$ ,  $G - X$  has at most  $2^{|X|}$  connected components.

- ▶ The only trees that are set graphs are paths.

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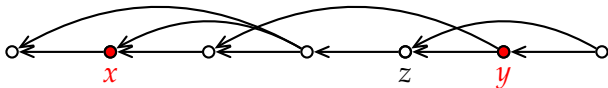




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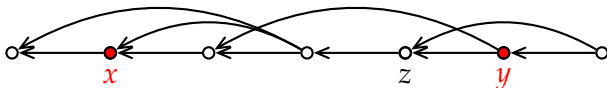
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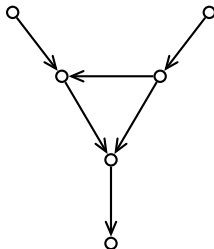
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Not all set graphs have a Hamiltonian path:



# RECOGNIZING SET GRAPHS IS NP-COMPLETE

Let  $S(G)$  be obtained from  $G$  by subdividing each edge once.

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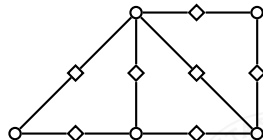
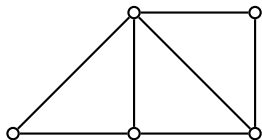
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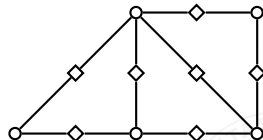
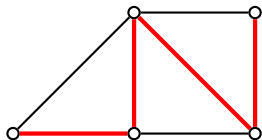
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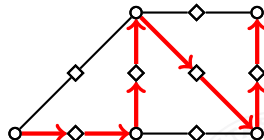
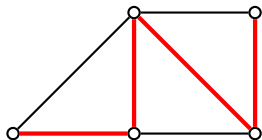
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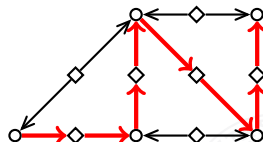
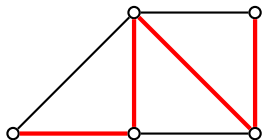
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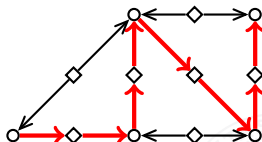
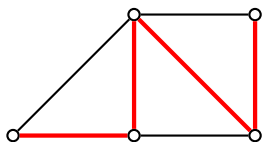
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" $\Leftarrow$ ": prove that any EA orientation of  $S(G)$  has a directed path passing through all vertices of  $G$ .



# MORE ON COMPLEXITY

An EAD is said to be

- ▶ **slim**, if the *removal* of any arc disrupts extensionality
- ▶ **dependent**, if the *reversal* of any arc disrupts extensionality or acyclicity



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Counting the EA, slim EA, dependent EA orientations of a given graph is #P-complete.



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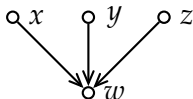
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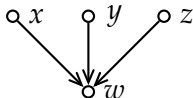
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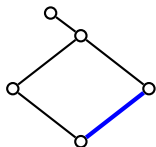


- ▶ use claw-freeness to 'resolve' each collision, by reversing at most two arcs

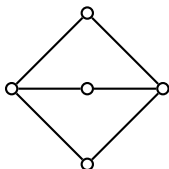
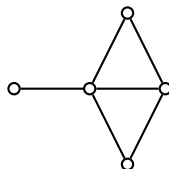




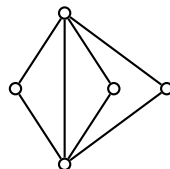
# MORE ON CLAW-FREENESS



apple

 $K_{2,3}$ 

dart

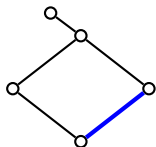
 $\text{co-}(K_3 + 2K_1)$ 

## THEOREM

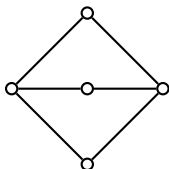
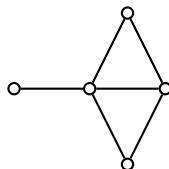
If  $G$  is  $(\text{apple}, K_{2,3}, \text{dart}, \text{co-}(K_3 + 2K_1))$ -free and connected, then

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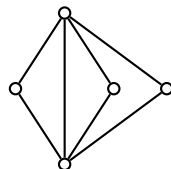
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- ▶ this is the largest hereditary class with this property

# GENERALIZED CLAWS AND $r$ -EXTENSIONALITY

What about forbidding  $K_{1,r+2}$ , for any  $r \geq 1$ ?

## DEFINITION

A digraph is said to be  **$r$ -extensional** if for any  $v \in V(D)$ ,

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## THEOREM

*A  $K_{1,r+2}$ -free graph  $G$  admits an  $r$ -extensional acyclic orientation  $\Leftrightarrow G$  has at most  $r$  connected components.*

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[Charbit, Charon, Cohen, Hudry, Lobstein, 2006, 2008]



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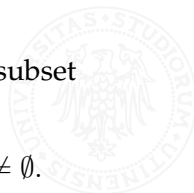
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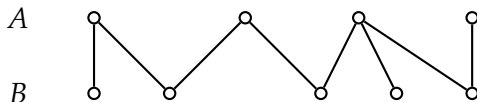
Given a bipartite graph  $G = (A \uplus B, E)$ , we say that a subset  $C \subseteq B$  is a **discriminating code (DC)** if

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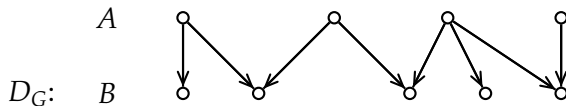




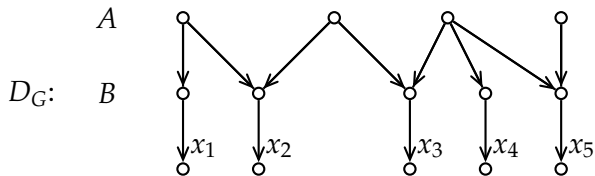
## FINDING A MINIMUM SIZE OOSC (2)



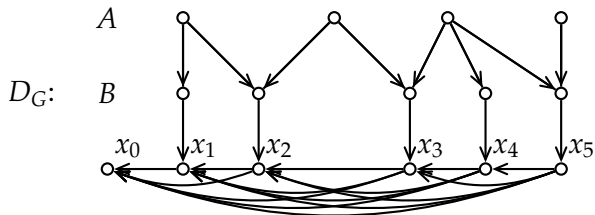
## FINDING A MINIMUM SIZE OOSC (2)



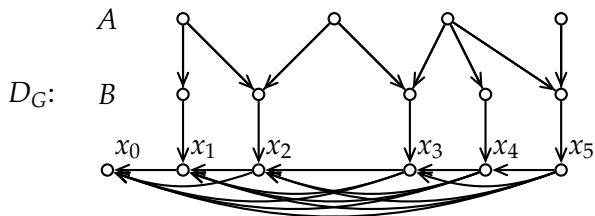
## FINDING A MINIMUM SIZE OOSC (2)



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## FINDING A MINIMUM SIZE OOSC (2)



- $G = (A \uplus B, E)$  has a DC  $C \subseteq B$  of size at most  $k \Leftrightarrow D_G$  has an ooSC of size at most  $k + |B|$ .



# CONCLUSIONS

- ▶ Set graphs arise from set theory, but have many (practical) applications
- ▶ Recognizing set graphs is NP-complete
- ▶ Set graphs generalize:
  - ▶ graphs with a Hamiltonian path
  - ▶ connected claw-free graphs (an EA orientation can be found in poly-time)
  - ▶ claw-freeness and set graphs are quite intertwined
- ▶ Connections between sets/set theory and identifying/separating codes.



MERSI!

