

Open neighborhood Locating-Dominating sets

OLD(G)

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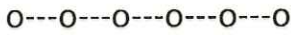
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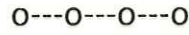
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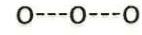
$M^{\text{IDNT}}(n) = \text{OLD}(Q_n)$: "identifying codes with non-transmitting faulty vertices"



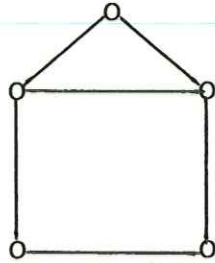
$$\text{OLD}(P_6) = 4$$



$$\text{OLD}(P_4) = 4$$



$$\text{OLD}(P_3)$$



$$\text{OLD}(H) = 3$$

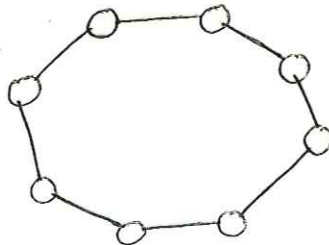
$S \subseteq V(G)$ is an open-locating-dominating (OLD) set if for each vertex w we have $N(w) \cap S \neq \emptyset$ and $w \neq x$ implies $N(w) \cap S \neq N(x) \cap S$.

The open-locating dominating number $\text{OLD}(G)$ is the minimum cardinality of an OLD(G)-set.

Theorem 1. G has an OLD(G)-set iff $\delta(G) \geq 1$ and $w \neq x$ implies $N(w) \neq N(x)$.

Theorem 2. If $\text{OLD}(G) = h$, then $|V(G)| \leq 2^h - 1$.

To see that this upper bound is achieved, let C_h be the cycle on $h \geq 5$ vertices, $V(C_h) = \{v_1, v_2, \dots, v_h\}$ and $E(C_h) = \{v_1v_2, v_2v_3, \dots, v_{h-1}v_h, v_hv_1\}$. Note that $N(v_i) = \{v_{i-1}, v_{i+1}\} \pmod{h}$, and the open neighborhoods $N(v_1), \dots, N(v_h)$ are distinct. We can add $2^h - 1 - h$ additional vertices, each of which has a distinct open neighborhood in $V(C_h)$, to obtain a graph G_h with $|V(G_h)| = 2^h - 1$ and $\text{OLD}(G_h) = h$.



THEOREM. Decision problem "Is $OLD(G) \leq K$?" is NP-complete.

As noted in Garey and Johnson, Problem 3-SAT is NP-complete.

3-SAT

INSTANCE: Collection $C = \{c_1, c_2, \dots, c_M\}$ of clauses on set $U = \{u_1, u_2, \dots, u_N\}$ such that $|c_i| = 3$ for $1 \leq i \leq M$.

QUESTION: Is there a satisfying truth assignment for C ?

Open-Locating-Dominating (OLD)

INSTANCE: Graph $G = (V, E)$ and positive integer $K \leq |V|$.

QUESTION: Is $OLD(G) \leq K$?

Theorem 3 [14]. Problem *OLD* is NP-complete.

Proof. Clearly *OLD* \in NP. We show a polynomial time reduction from 3-SAT to *OLD*. Given U and C , for each u_i construct the graph G_i on 21 vertices illustrated in Figure 2(a), and for each clause c_j construct the graph H_j illustrated in Figure 2(b). Finally, to complete the construction of graph G , for $1 \leq j \leq M$ if clause $C_j = \{u_{j,1}, u_{j,2}, u_{j,3}\}$ where each $u_{j,t}$ is some u_i or \bar{u}_i , let clause vertex c_j be adjacent to variable vertices $u_{j,1}, u_{j,2}$ and $u_{j,3}$. Note that G has $21N + 7M$ vertices and $27N + 10M$ edges and can be constructed from C in polynomial time.

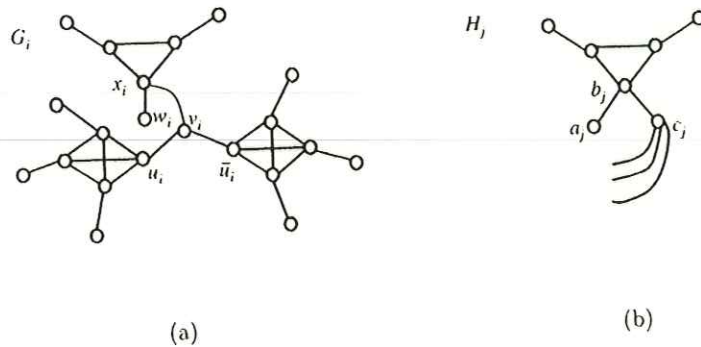


Figure 2: Variable and clause graphs G_i and H_j , $1 \leq i \leq N$, $1 \leq j \leq M$

If $S \subseteq V(G)$ is any *OLD*(G)-set, then S must contain every vertex adjacent to an endpoint because S is open-dominating. That is, S contains the $9N+3M$ darkened vertices of Figure 2. Because $N(w_i) \cap S = \{x_i\} \neq N(v_i) \cap S$, it must be the case that $(N(v_i) \cap S) \cap \{u_i, \bar{u}_i\} \neq \emptyset$. Likewise, $N(a_j) \cap S = \{b_j\} \neq N(c_j) \cap S$, so $S \cap \{u_{j,1}, u_{j,2}, u_{j,3}\} \neq \emptyset$.

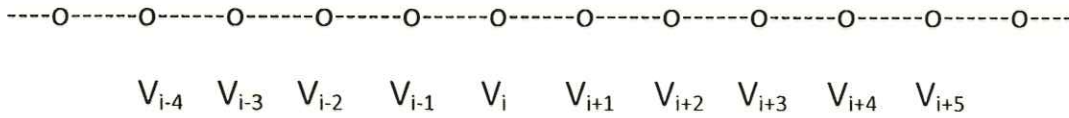
It is now easy to see that C has a satisfying truth assignment if and only if

$$OLD(G) = (9N + 3M) + N = 10N + 3M. \square$$

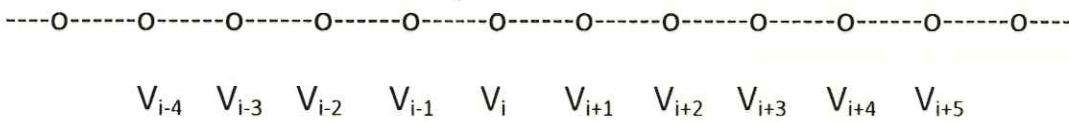


PATH P_N and CYCLE C_N

- $S \cap \{V_{i-1}, V_{i+1}\} \neq \emptyset$.
- Assume $S \cap \{V_i, V_{i+1}\} = \emptyset$:



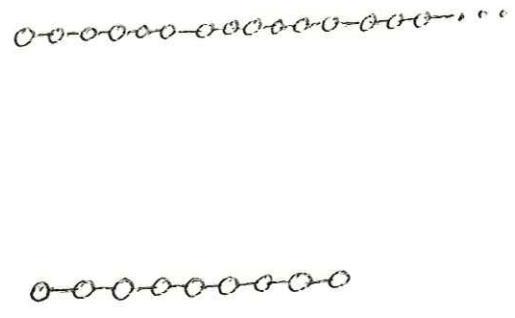
- Assume $\{V_{i-1}, V_{i+1}\} \subset S$ and $V_i \notin S$:



- If S is an OLD(P_n)-set for $n \geq 5$, then $\{v_2, v_4\} \subset S$ and $|S \cap \{v_1, v_3, v_5\}| = 2$.

→ $OLD(P_n) \geq (2/3)n$. $\text{o---o---o---o---o---o---o---o---o---o---}$

N	OLD(P_N)	OLD(C_N)
6k	4k	4k
6k+1	4k+1	4k+1
6k+2	4k+2	4k+2
6k+3	4k+3 *	4k+2
6k+4	4k+4 *	4k+4 *
6k+5	4k+4	4k+4



$v \in \text{OLD}(P_n)\text{-set } S \rightarrow \text{sh}^0(v; S) \leq 1 + \frac{1}{2} = \frac{3}{2}$

→ $|S| \geq n / (3/2) = (2/3)n$

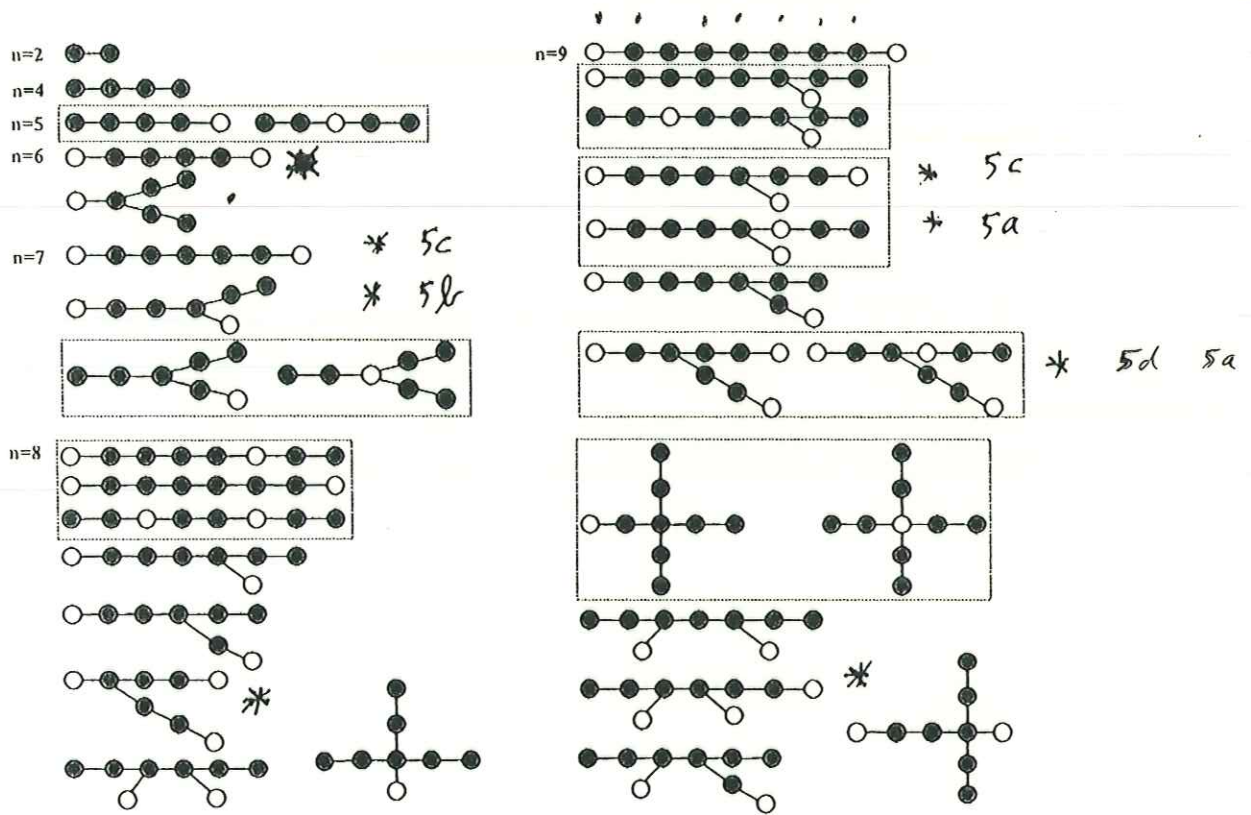


Fig. 1. All trees $T \in \mathfrak{S}$ of order $n \leq 9$.

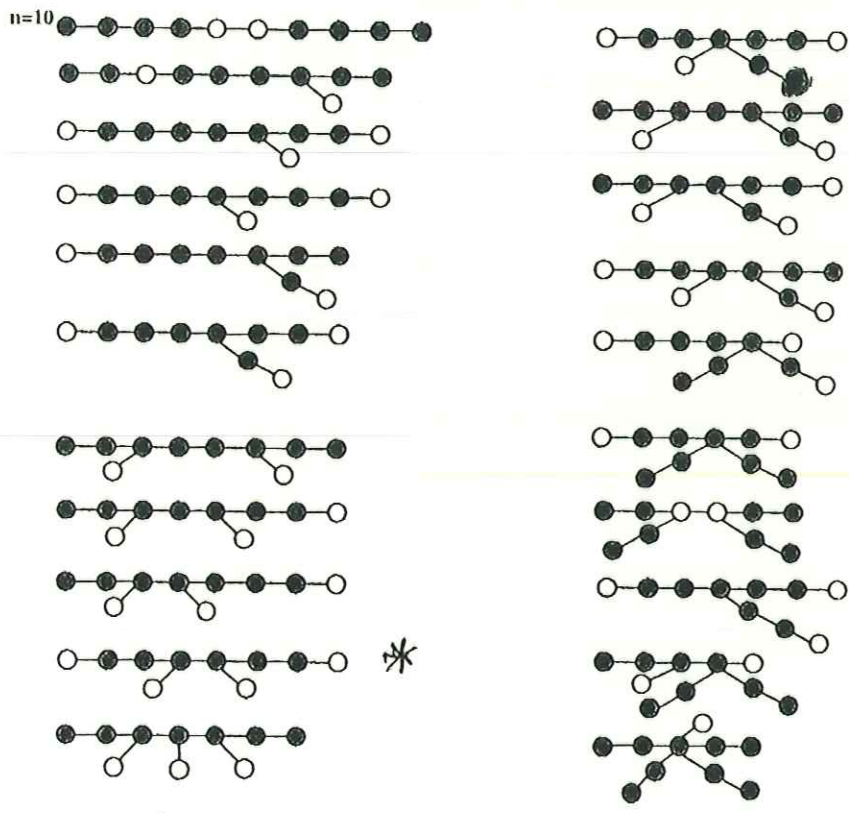


Fig. 2. All trees $T \in \mathcal{S}$ of order $n = 10$.

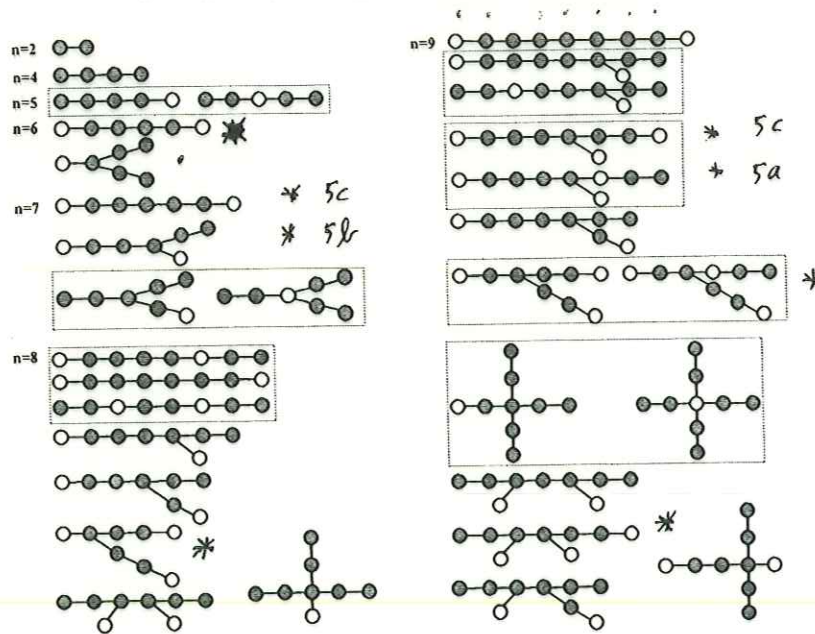


Fig. 1. All trees $T \in \mathfrak{S}$ of order $n \leq 9$.

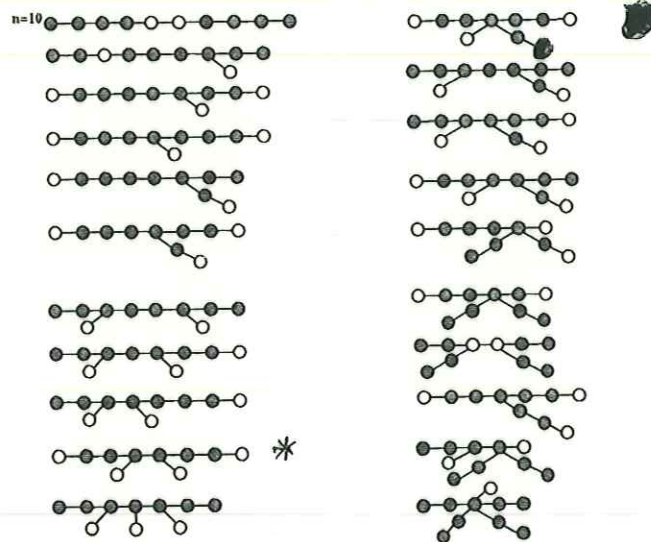


Fig. 2. All trees $T \in \mathfrak{S}$ of order $n = 10$.

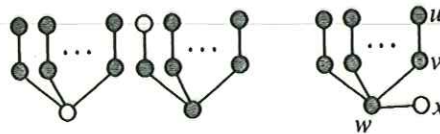


Fig. 3. $OLD(T_n)$ -sets where $OLD(T_n) = n - 1 \geq 4$.

We will say that vertex set $S \subseteq V(G)$ open-distinguishes vertex u if $\emptyset \neq N(u) \cap S \neq N(x) \cap S$ for all $x \in V(G) - u$. In particular, an OLD-set for G must open-distinguish every vertex.

Observation 5. (i) If u is a support vertex of an endpoint $v \in V(G)$, then every OLD-set dominates v , so u is in every OLD-set S of G . (ii) If u, v, w, x is a path in a graph G with degrees $\deg u = \deg x = 1$ and $\deg v = 2$ (as in Fig. 3) and S is an OLD-set, then $u \in S$ (or else $N(v) \cap S = N(x) \cap S$) as well as support vertices v and w . (iii) If the girth of G satisfies $g(G) \geq 5$ (in particular, if G is a tree) and $|N(v) \cap S| \geq 2$, then S open-distinguishes v .

Theorem 6. For $n \geq 5$ there is a unique tree T_n of order n with $OLD(T_n) = n - 1$. If $n = 2k$, then T_n is obtained by subdividing all but one of the edges of the star $K_{1,k}$, and the $OLD(T_{2k})$ -set is unique. If $n = 2k + 1$, then T_n is the subdivision graph of $K_{1,k}$, and there are two essentially different $OLD(T_{2k+1})$ -sets as illustrated in Fig. 3.

Proof. Clearly the trees T in Fig. 3 (of diameter four) satisfy $OLD(T) = |V(T)| - 1$ with the $OLD(T)$ -sets precisely as indicated. If $T \in \mathfrak{S}$ and $\text{diam}(T) \leq 3$, then T must be P_2 or P_4 . If $T \in \mathfrak{S}$ and $\text{diam}(T) \geq 5$, let u, v, w, \dots, a, b, c be a diametric path (necessarily, $\deg v = \deg b = 2$). If w is the support vertex of a vertex x , let $v_1 = x$. Otherwise, let $v_1 = u$. Likewise, if an



TREES

$\mathcal{T} = \{ T_n : \text{no two endpoints have the same support vertex} \} \quad (l = s)$

Theorem (Slater, 1987) $(1/3)n < LD(T_n) \leq n-1$

(Blidia, et al 2007) $(n+l-s+1)/3 \leq LD(T_n) \leq (n+l-s)/2$

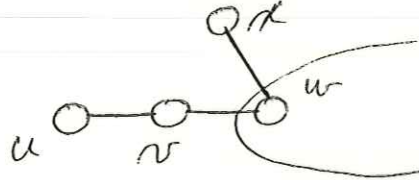
(Bertrand, et al 2005) $3(n+1)/7 \leq IC(T_n)$

(Blidia, et al 2007) $3(n+l-s+1)/7 \leq IC(T_n)$

(PS and Sewell, 2011+) $IC(T_n) \leq (2/3)(n-1/2) + l-s$

OBSERVATIONS: Let S be an OLD(T)-set.

- (i) u a support vertex of endpoint $v \rightarrow u \in S$
- (ii) path u, v, w, x with $\deg u = \deg x = 1$ and $\deg v = 2 \rightarrow u \in S$



- (iii) $|N(u) \cap S| \geq 2 \rightarrow S$ open-distinguishes u .



Theorem(Seo and PS, 2010) $(n/2) + 1 \leq \text{OLD}(T_n) \leq n-1$ for $T \in \mathcal{T}$ and $n \geq 5$.

Proof. ($\leq n-1$)

($\text{OLD}(T_n) = n-1$ iff $\text{diam}(T) = 4$)

CLAIM: $OLD(T_N) \geq \lceil N/2 \rceil + 1$

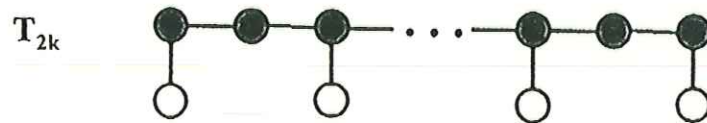
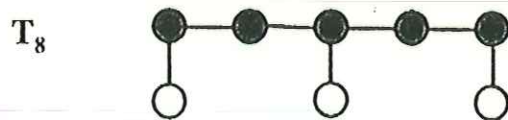
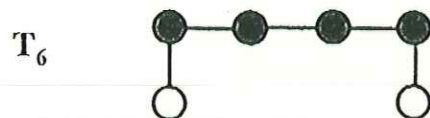


Fig. 4. The unique tree T_{2k} of order $n = 2k \neq 4$ with $OLD(T_{2k}) = k + 1 = \lceil n/2 \rceil + 1$.

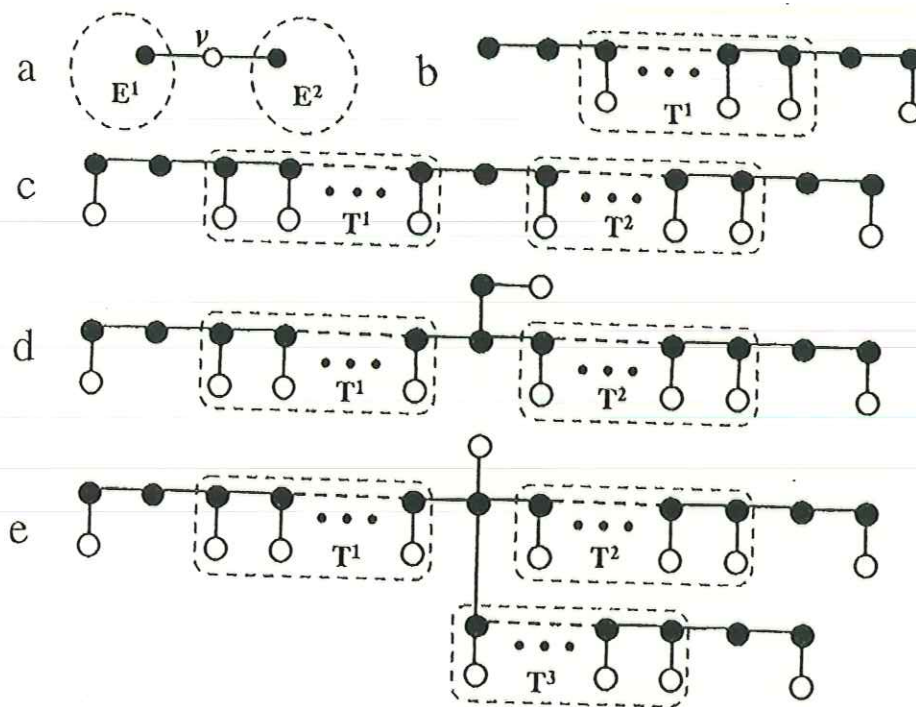


Fig. 5. $OLD(T_{2k+1}) = k + 2$. Each T^l is a (possibly empty) comb.

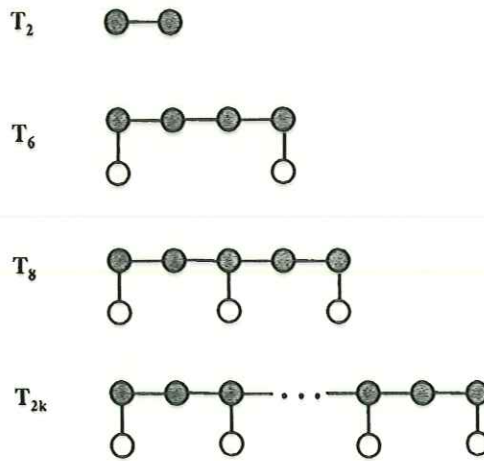


Fig. 4. The unique tree T_{2k} of order $n = 2k \neq 4$ with $OLD(T_{2k}) = k + 1 = \lceil n/2 \rceil + 1$.

Then,

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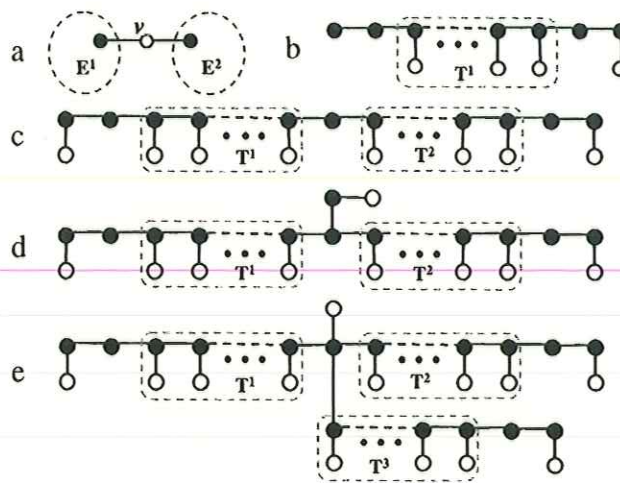


Fig. 5. $OLD(T_{2k+1}) = k + 2$. Each T^i is a (possibly empty) comb.

Lemma 13. If $OLD(T_{2k}) = k + 1 \geq 4$, then T_{2k}^* is a path on $k + 1$ vertices.

Proof. If x is an endpoint of T_{2k}^* with support vertex $t \in V(T_{2k}^*)$, then t cannot be the support vertex in T_{2k} of an endpoint y of T_{2k} or else (as in the proof of Lemma 11), $N(y) \cap S = \{t\} = N(x) \cap S$, a contradiction. Also, no two endpoints x_1 and x_2 of T_{2k}^* can have a common neighbor t or else $N(x_1) \cap S = \{t\} = N(x_2) \cap S$, a contradiction. Now if T_{2k}^* has a vertex of degree at least three, then T_{2k}^* would have three endpoints with distinct support vertices that are not support vertices in T_{2k} , a contradiction because, as noted, there are exactly two vertices in $V(T_{2k}^*)$ that are not support vertices of T_{2k} . \square

Thus we have the following theorem.

Theorem 14. If $OLD(T_{2k}) = k + 1 \geq 4$, then T_{2k}^* is a path $v_1, v_2, \dots, v_k, v_{k+1}$ where v_i is a support vertex of T_{2k} if and only if $i \notin \{2, k\}$. The only other tree of even order achieving lower bound $OLD(T_n) = n/2 + 1$ is P_2 .

For odd values of n with $OLD(T_n) = \lceil n/2 \rceil + 1$ one might or might not have all of the interior vertices in an $OLD(T_n)$ -set S . Note that in Fig. 1 there are three trees of order 9 with OLD -sets of order 6. Two of these have OLD -sets of both types, while the third has all of its interior vertices in its unique OLD -set S . We first show that at most one interior vertex will not be in S .

Lemma 15. If $OLD(T_n) = \lceil n/2 \rceil + 1$, then, for $OLD(T_n)$ -set S , at most one interior vertex is not in S .

Proof. Assume that v_1 and v_2 are distinct interior vertices not in S . By Proposition 10, n is odd and $deg v_1 = deg v_2 = 2$. Let T^1 (respectively, T^2) be the subtree that is the component of $T_n - v_1$ (respectively, $T_n - v_2$) that does not contain v_2 (respectively, v_1). Let T^3 be the third component of $T_n - v_1 - v_2$. By Proposition 10, $|V(T^1)|$ and $|V(T^2)|$ are even, and so $|V(T^3)|$ is odd. Let $S_i = S \cap V(T^i)$ for $1 \leq i \leq 3$. Then $v_1, v_2 \notin S$ implies that S_i is an OLD -set for T^i . Now $|S_1| \geq n_1/2 + 1$, $|S_2| \geq n_2/2 + 1$, and $|S_3| \geq (n_3 + 1)/2 + 1$. Thus, $|S| \geq (n_1 + n_2 + n_3 + 2)/2 + 5/2 = \lceil n/2 \rceil + 2$, a contradiction. \square

Following from Proposition 10 and Lemma 15, we have the following theorem.

Theorem 16. If $OLD(T_{2k+1}) = k + 2$ and S is an $OLD(T_{2k+1})$ -set with interior vertex $v \notin S$, then T_{2k+1} is as in Fig. 5(a) where trees E^1 and E^2 are even order trees on n_1 and n_2 vertices with $OLD(E^i) = n_i/2 + 1$, and the neighbors of v are in S . (See Theorem 14 and Fig. 4.)

It remains to characterize the odd order trees T_{2k+1} with an $OLD(T_{2k+1})$ -set S of order $k + 2$ in which every interior vertex is in S , $V(T_{2k+1}^*) \subset S$. There are two cases to consider depending on whether or not S contains an endpoint of T

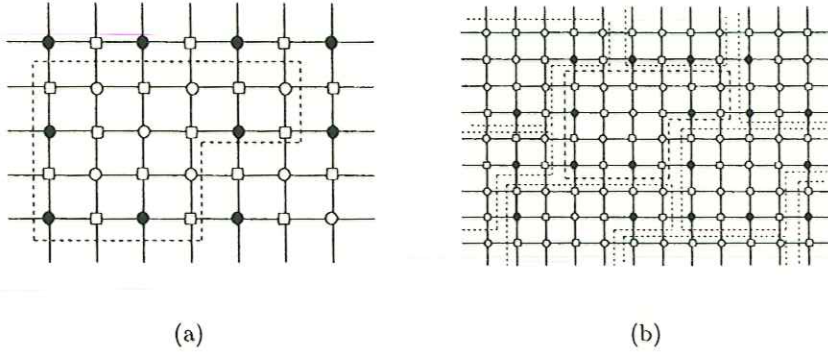


Figure 5: An $OLD(Z \times Z)$ - tiling

For the infinite grid $Z \times Z$ that is regular of degree 4, by Theorem 12, $OLD\%(Z \times Z) \geq 2/(1 + 4) = 2/5$. To show that $OLD\%(Z \times Z) \leq 2/5$, we observe that $Z \times Z$ is bipartite and partition $V(Z \times Z)$ into two groups, say square vertices and round vertices as shown Figure 5(b). If we use 4/10 of the round vertices by repeating the tile pattern shown in Figures 5(a,b), all of the square vertices are open-dominated and located. Similarly 2/5 of the square vertices can open-dominate and locate the round vertices, so we have $OLD\%(Z \times Z) \leq 2/5$ which results in the following theorem.

Theorem 13. For infinite square grid $Z \times Z$, $OLD\%(Z \times Z) = 2/(1 + 4) = 2/5$.

For the infinite hexagonal grid HX that is regular of degree 3, by Theorem 12, $OLD\%(HX) \geq 2/(1 + 3) = 1/2$. To show that $OLD\%(HX) \leq 1/2$, we observe that HX is bipartite and partition $V(HX)$ into square vertices and round vertices. The set containing 1/2 of the round vertices, as shown in Figure 6(b), dominates and locates all of the square vertices. Similarly 1/2 of the square vertices can open-dominate and locate the round vertices, so we have $OLD\%(HX) \leq 1/2$ which results in the following theorem.

Theorem 14. For infinite hexagonal grid HX , $OLD\%(HX) = 2/(1 + 3) = 1/2$.

For the infinite triangular grid TR that is regular of degree 6, by Theorem

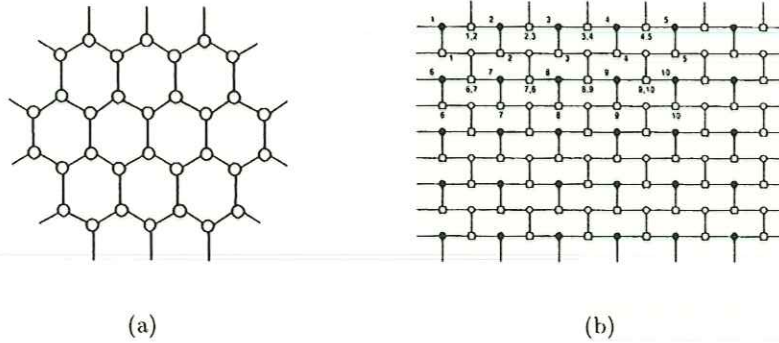


Figure 6: Infinite hexagonal grid HX

12, $OLD\%(TR) \geq 2/(1 + 6) = 2/7$. The set of darkened vertices shown in Figure 7 is an OLD -set and its cardinality is $1/3$, so we have $OLD\%(TR) \leq 1/3$ which results in the following theorem. To date, we have not determined the exact value for $OLD\%(TR)$.

Theorem 15. *For the infinite triangular TR which is regular of degree six, we have $2/7 \leq OLD\%(TR) \leq 1/3$.*

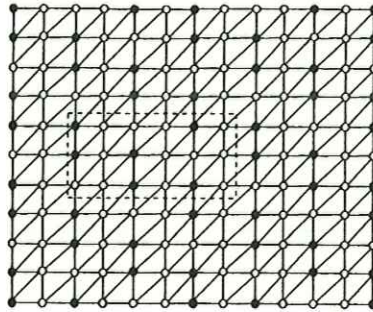
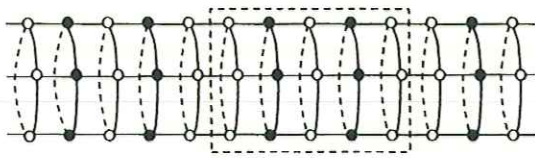


Figure 7: Infinite triangular grid TR

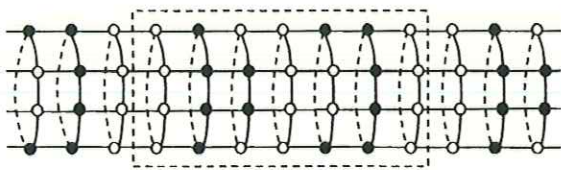


2. INFINITE CYLINDERS

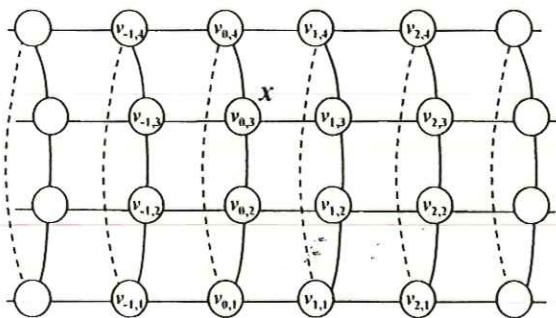
To illustrate these concepts, consider the infinite cylindrical graphs $P_{\infty} \square C_k$ for infinite path P_{∞} and cycle C_k , as shown in Figure 2 for $k = 3$ and 4. Because the 4-cycle C_4 does not have an open-locating set (in particular, $V(C_4)$ is not open-locating), we have a special case for the infinite cylinders.



(a)



(b)



(c)

Figure 2. $P_{\infty} \square C_k$

Theorem 3. For $k = 3$ and $k \geq 5$ we have $OLD\%(P_{\infty} \square C_k) = 2/5$, and $OLD\%(P_{\infty} \square C_4) = 3/7$.

Proof. When $k \neq 4$, we can use every second and fourth cycle in each successive set of five cycles as in Figure 2(a). Specifically, $D = \{v_{i,j} \mid 1 \leq j \leq k \text{ and } i \equiv 2 \text{ or } 4 \pmod{5}\}$ is open-locating. Hence, $OLD\%(P_{\infty} \square C_k) \leq 2/5$. By Theorem 2, $OLD\%(P_{\infty} \square C_k) \geq 2/5$.

Now, consider $P_{\infty} \square C_4$. Figure 2(b) shows an OLD -set using $3/7$ of the vertices of $V(P_{\infty} \square C_4)$, hence $OLD\%(P_{\infty} \square C_4) \leq 3/7$. To see $OLD\%(P_{\infty} \square C_4) \geq 3/7$, we will show that for all vertices x in an open-dominating set D , $sh^o(x; D) \leq 7/3$.

By Theorem 1 $sh^o(x; D) \leq 5/2 = 1 + 1/2 + 1/2 + 1/2$, and to achieve $sh^o(x; D) = 5/2$, vertex x has to have a private neighbor and the other neighbors of x have to be dominated exactly twice. Let x be $v_{0,3}$ as shown in Figure 2(c). Suppose the private neighbor of x is on the C_4 , say $v_{0,4}$. Then, $v_{-1,4}, v_{1,4}, v_{0,1} \notin D$ and $v_{0,2}$ has to be dominated by either $v_{-1,2}$ or $v_{1,2}$. Without loss of generality assume that $v_{1,2} \in D$. Then, $N(v_{0,2}) \cap D = N(v_{1,3}) \cap D = \{x, v_{1,2}\}$, a contradiction. On the other hand, if the private neighbor of x is on the path, say $v_{1,3}$, then $v_{1,4}, v_{2,3}, v_{1,2} \notin D$ and $v_{0,2}$ has to be dominated by either $v_{-1,2}$ or $v_{0,1}$. If $v_{-1,2} \in D$, then $N(v_{1,3}) \cap D = N(v_{0,2}) \cap D = \{x, v_{-1,2}\}$, a contradiction, and if $v_{0,1} \in D$, then $N(v_{0,2}) \cap D = N(v_{0,4}) \cap D = \{x, v_{0,1}\}$, a contradiction as well. Therefore, $sh^o(x; D) < 5/2 = 1 + 1/2 + 1/2 + 1/2$, which means that $sh^o(x; D) \leq 1 + 1/2 + 1/2 + 1/3 = 7/3$. \square