



Structural properties of identifying codes: some results, conjectures and open problems

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with D. Auger, I. Charon, I. Honkala, A. Lobstein



Synthesis of five articles dealing with structural properties of identifying codes

- I. Charon, I. Honkala, O. Hudry, A. Lobstein, Structural Properties of Twin-Free Graphs, *Elec. J. of Comb.* 14 (1), R 16, 2007.
- D. Auger, Induced paths in twin-free graphs, *Elec. J. of Comb.* 15 (1), N 17, 2008.
- D. Auger, I. Charon, O. Hudry, A. Lobstein, On the existence of a cycle of length at least 7 in a $(1, \leq 2)$ -twin-free graph, *Discussiones Mathematicae-Graph Theory* 30, 2010, 591-609.
- I. Charon, O. Hudry, A. Lobstein, Minimum sizes of identifying codes in graphs differing by one vertex, submitted.
- I. Charon, O. Hudry, A. Lobstein, Minimum sizes of identifying codes in graphs differing by one edge, submitted.

Definitions and notation

- $G = (X, E)$ will denote an undirected graph, n will denote the number of vertices of G and r will denote a positive integer.
- For x in X , $B_r(x)$ denotes the *ball* of radius r and centre x :

$$B_r(x) = \{y \in X \text{ s.t. } d(x, y) \leq r\},$$

where $d(x, y)$ is the length of a shortest path between x and y in G .

- Two vertices x and y are said to be *r-twins* if they own the same ball: $B_r(x) = B_r(y)$. If there is no *r-twins* in G , then G is said to be *r-twin-free*.

- A subset C of G is an *r-identifying code* (*r-IC*) of G if:

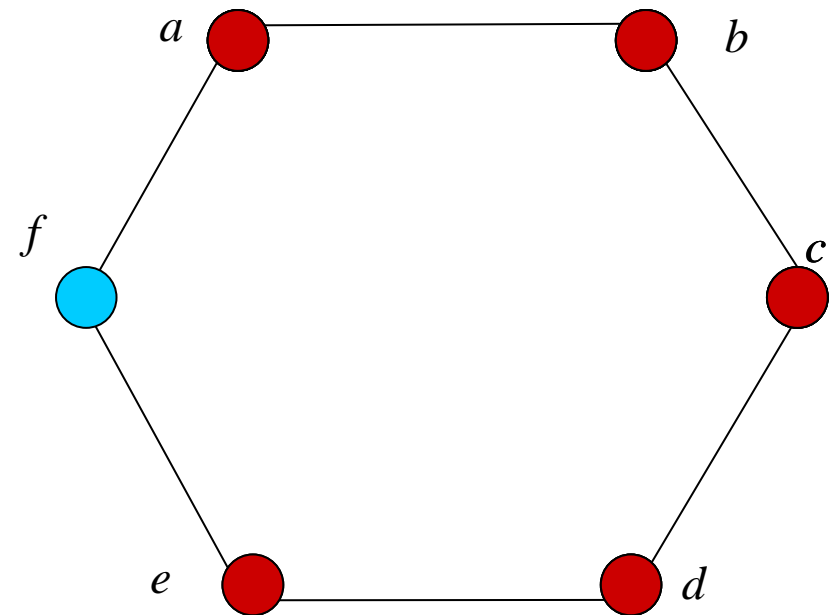
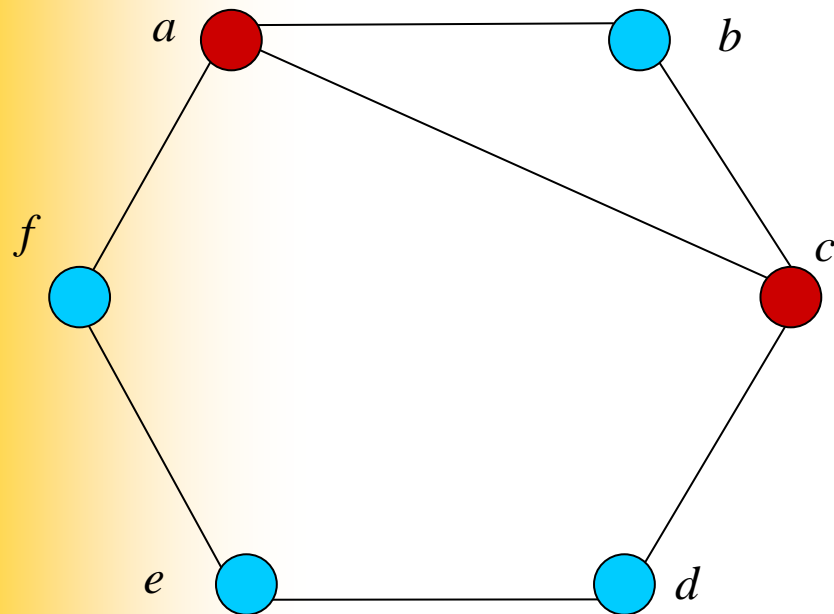
$\forall (x, y) \in X^2$ with $x \neq y$, $\emptyset \neq B_r(x) \cap C \neq B_r(y) \cap C \neq \emptyset$.

Then G is said to be *r-identifiable*.

- **Theorem:** G is *r-identifiable* if and only if G is *r-twin-free*. In this case, $C = X$ is an *r-IC* of G .
- **Problem:** when G is *r-identifiable*, determine an *r-IC* of G of minimum size.
- The size of a minimum *r-IC* of G is denoted by $M_r(G)$.

Example

- On the left, a and c are 2-twins: $B_2(a) = B_2(c)$.



- On the right, $C = \{a, b, c, d, e\}$ is a minimum 2-IC: $M_r(C_6) = 5$.

Generalization to $(r, \leq l)$ -identifying codes

- Given a graph $G = (X, E)$ and a subset Y of X , we set:

$$B_r(Y) = \cup \{B_r(y) \text{ for } y \in Y\}.$$

- A subset C of X is said to be an $(r, \leq l)$ -*identifying code* if we have, for any pair of distinct subsets Y and Z with $|Y| \leq l$ and $|Z| \leq l$:

$$\emptyset \neq B_r(Y) \cap C \neq B_r(Z) \cap C \neq \emptyset.$$

- Two subsets Y and Z are said to be $(r, \leq l)$ -*twins* if we have $|Y| \leq l$, $|Z| \leq l$, $Y \neq Z$ and $B_r(Y) = B_r(Z)$. If there is no $(r, \leq l)$ -twins in G , then G is said to be $(r, \leq l)$ -*twin-free*.

Outline

I. Unavoidable subgraphs in twin-free graphs

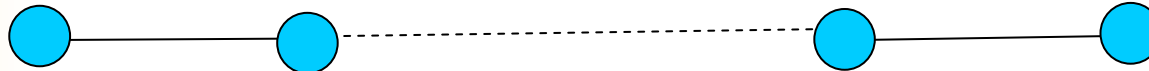
II. r -terminal graphs

III. Variations of $M_r(G)$ when a vertex is deleted
(or added)

IV. Variations of $M_r(G)$ when an edge is deleted
(or added)

I. Unavoidable subgraphs in twin-free graphs

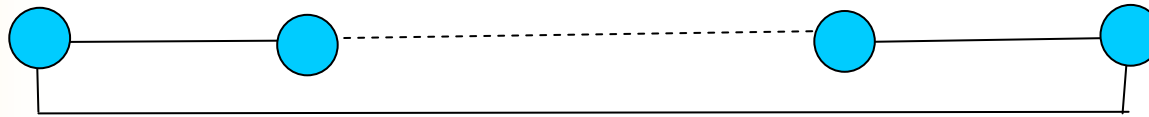
- Let P_k denote the path with k vertices:



- Theorem:** any r -twin-free graph with at least one edge contains P_{2r+1} as an **induced** subgraph.
- Corollary** (J. Moncel, PhD thesis, 2005): any r -twin-free graph with at least one edge contains at least $2r + 1$ vertices.
- In fact, P_{2r+1} is the unique r -twin-free connected graph with $2r + 1$ vertices.

What about unavoidable subgraphs in $(1, \leq 2)$ twin-free graphs?

- Let C_k denote the cycle with k vertices:

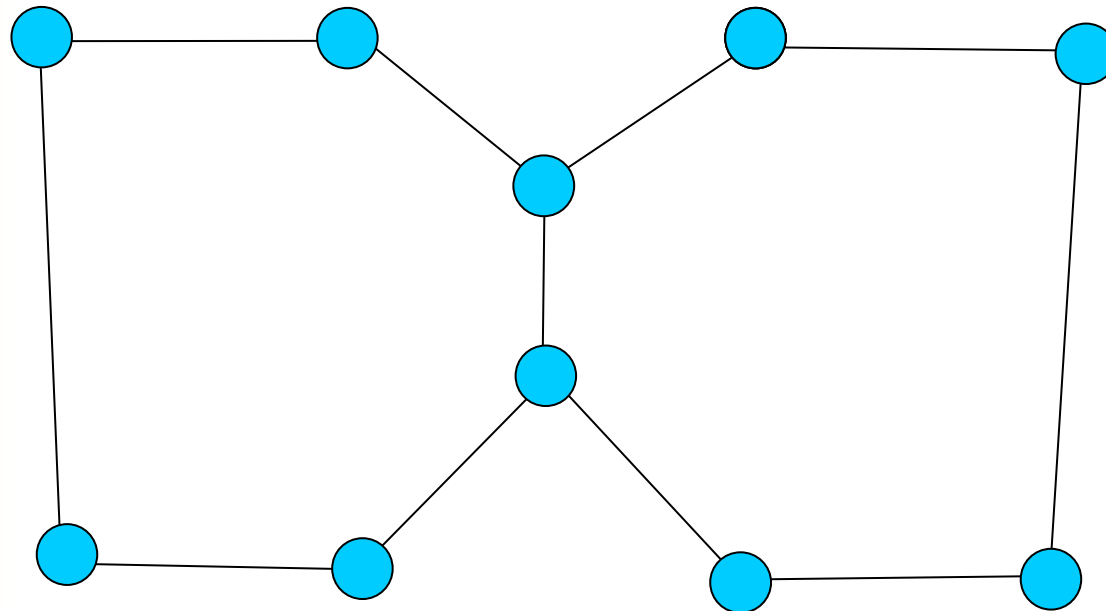


It is said to be *elementary* if it does not go through a same vertex twice.

- **Theorem:** any $(1, \leq 2)$ -twin-free graph with at least two vertices contains an elementary cycle C_k with $k \geq 7$ (not necessarily as an induced subgraph).
- **Corollary:** any $(1, \leq 2)$ -twin-free graph contains at least 7 vertices.

Example of a $(1, \leq 2)$ twin-free graph

- The graph below is $(1, \leq 2)$ -twin-free and contains C_{10} as a subgraph, but its largest chordless cycle is C_6 .



- **Conjectures.**

1. For any integer $r \geq 2$, the smallest connected $(r, \leq 2)$ -twin-free graph with at least two vertices is the cycle C_{4r+3} .

2. Any $(r, \leq 2)$ -twin-free graph with at least two vertices contains a cycle of length at least $4r + 3$.

- **Open problem:** what about similar results for $(r, \leq l)$ -twin-free graphs with $l > 2$?

- **Remark:** T. Laihonon (EuJC, 2008) gives a connected $(1, \leq 3)$ -twin-free graph with 16 vertices.

II. r -terminal graphs

- Let $G = (X, E)$ be a connected r -twin-free graph with at least 2 vertices; G is said to be r -terminal if, for any vertex x of G , $G_x = G - \{x\}$ is not r -twin-free.

Or: G is *not* r -terminal if there exists a vertex x of G such that G_x is r -twin-free.

- For instance, P_{2r+1} is r -terminal for any $r > 1$.
- **Problem:** what are the r -terminal graphs? For $r > 1$, is P_{2r+1} the only r -terminal graph?

The case $r = 1$

- **Theorem:** there is no 1-terminal graph.

The proof is based on the following theorem:

- **Theorem** (N. Bertrand, 2001; S. Gravier, J. Moncel, 2007; I.C., O.H., A.L., 2007; ...): let G be any connected 1-twin-free graph with $n \geq 3$ vertices.

Then $M_1(G) \leq n - 1$.

- **Remark:** the graphs G with $M_1(G) = n - 1$ are all known (F. Foucaud, E. Guerrini, M. Kovše, R. Naserasr, A. Parreau, P. Valicov, EuJC 2011).

There does not exist 1-terminal graph: proof

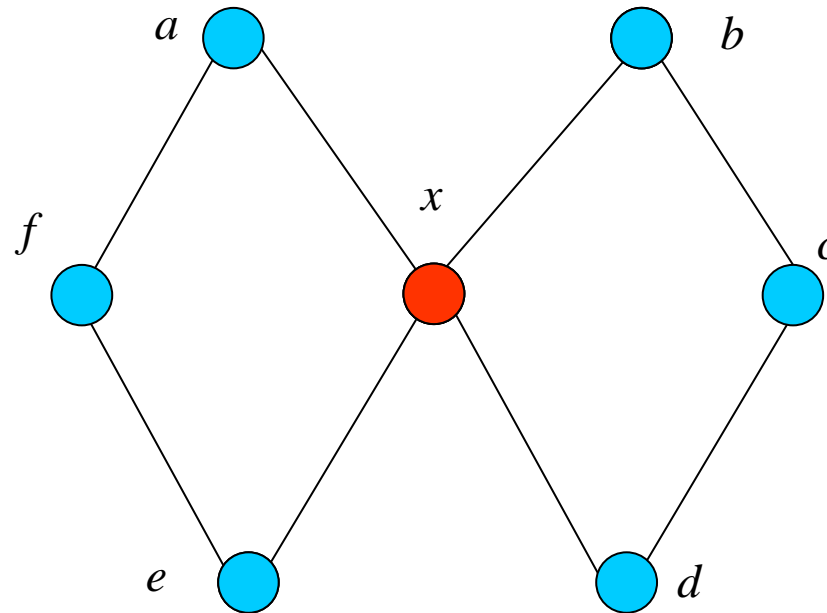
- If $n = 3$, then $G = P_3$, which is not terminal since we may remove the middle vertex while keeping a 1-twin-free graph:



- If $n > 3$, let C be a 1-IC of size $n - 1$ and choose x s.t. $X - C = \{x\}$. Then C is still a 1-IC of G_x , because removing x does not cut connexions between pairs of vertices not containing x . Thus G_x is 1-twin-free.

Example

- The graph below is 1-twin-free and $C = X - \{x\}$ is a 1-IC. Then G_x is 1-twin-free (though not connected).



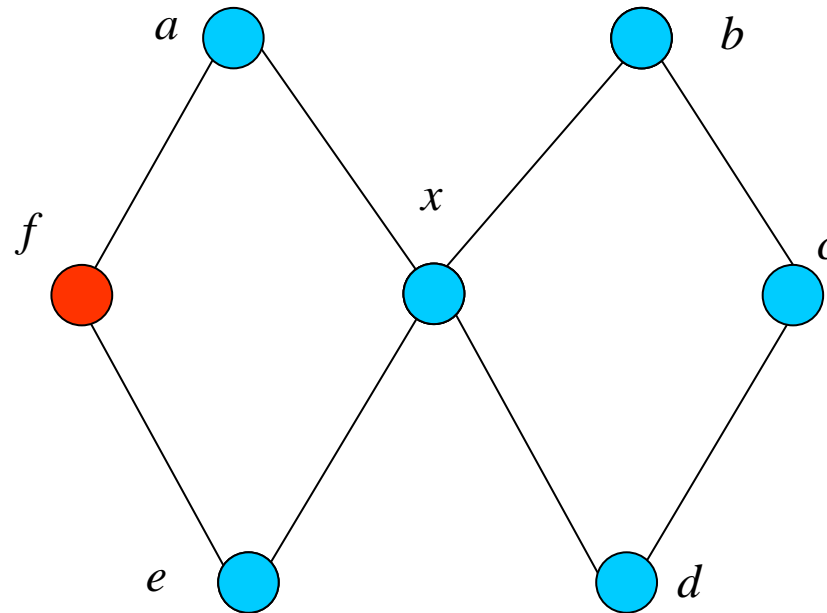


More precisely, we have:

- **Theorem:** Let $n \geq 4$ and G be any connected 1-twin-free graph with n vertices. Then there is a vertex x such that G_x is 1-twin-free and connected.

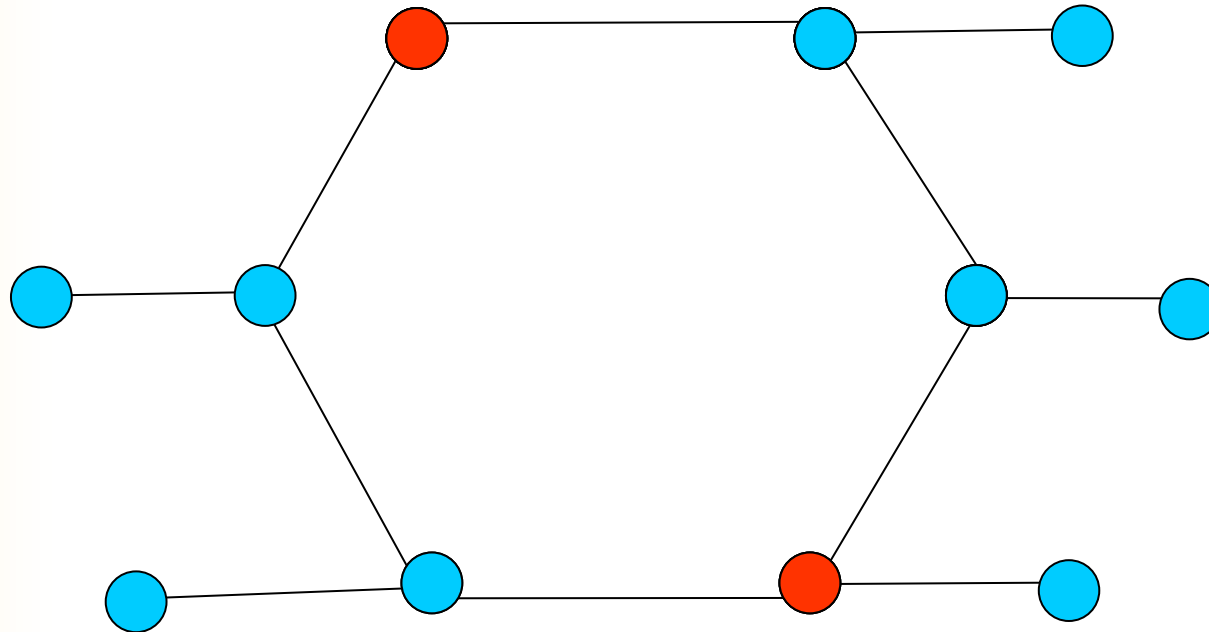
Example

- The graph below is 1-twin-free and $C = X - \{f\}$ is a 1-IC. Then G_f is 1-twin-free and connected.



The case $r \geq 3$

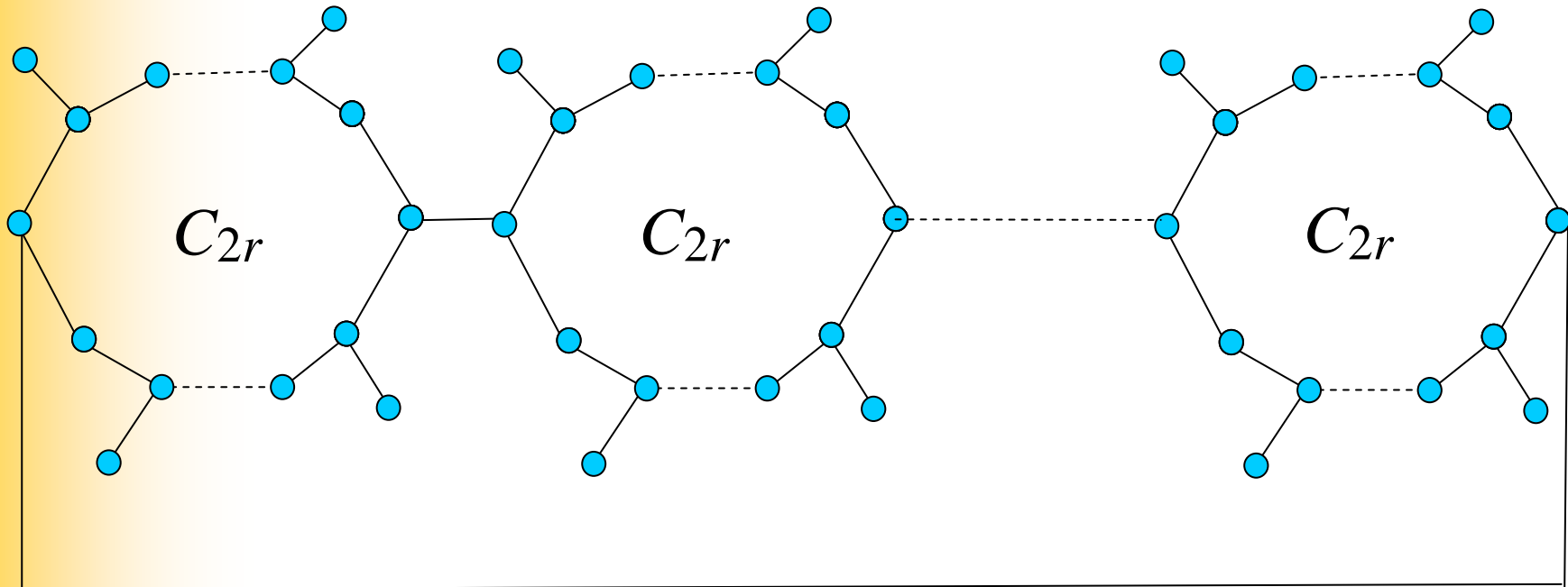
- **Theorem:** for each integer $r \geq 3$, there is a graph G which is r -terminal with $G \neq P_{2r+1}$.
- Ex.: $r = 3, n = 4r - 1$.



- Question: is there a finite number of such graphs?

The case $r \geq 6$

- **Theorem:** for each integer $r \geq 6$, there are infinitely many r -terminal graphs G .



Open problems

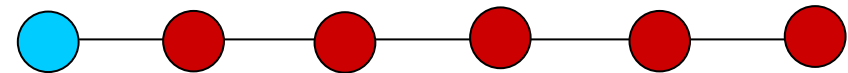
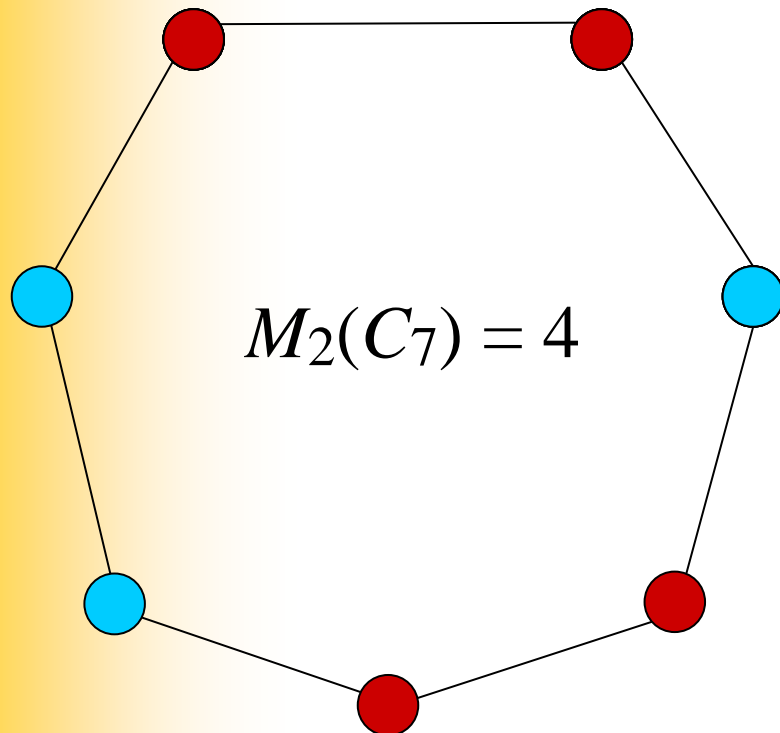
- For $r = 2$, is P_5 the only 2-terminal graph?
- For $3 \leq r \leq 5$, there exist r -terminal graphs, but are they in finite or infinite number?


The case of trees

- **Theorem:** let $r \geq 1$ and $n \geq 2r + 2$ be integers, and T be any r -twin-free tree with n vertices. Then there exists a leaf x s.t. $T_x = T - \{x\}$ is r -twin-free.
- **Corollary:**
 1. there is no 1-terminal tree;
 2. for $r > 1$, the only r -terminal tree is P_{2r+1} .

III. Variations of $M(G)$ when a vertex is added or deleted

- D. Roberts and F. Roberts (EuJC 2008) observed that removing a vertex from a cycle may increase M_2 by 1.



- 
-
- **Problem:** what are the maximum variations of $M_r(G)$ when a vertex is added or deleted?
 - More specifically, for any vertex x of G , we set $G_x = G - \{x\}$. We study the quantities $|M_r(G) - M_r(G_x)|$ and, when appropriate, $M_r(G)/M_r(G_x)$ (or, equivalently, $M_r(G_x)/M_r(G)$).

The case $r = 1$

Removing a vertex cannot involve a large decrease of M_1 :

- **Theorem:** Let G be any 1-twin-free graph. For any vertex x of G s.t. G_x is 1-twin-free, we have:

$$M_1(G_x) \geq M_1(G) - 1$$

(see also F. Foucaud, E. Guerrini, M. Kovše, R. Naserasr, A. Parreau, P. Valicov, EuJC 2011 for a stronger result).

The case $r = 1$

Removing a vertex may involve a large increase of M_1 :

- **Theorem:** There exists a (connected) graph G with n vertices and a vertex x of G with:

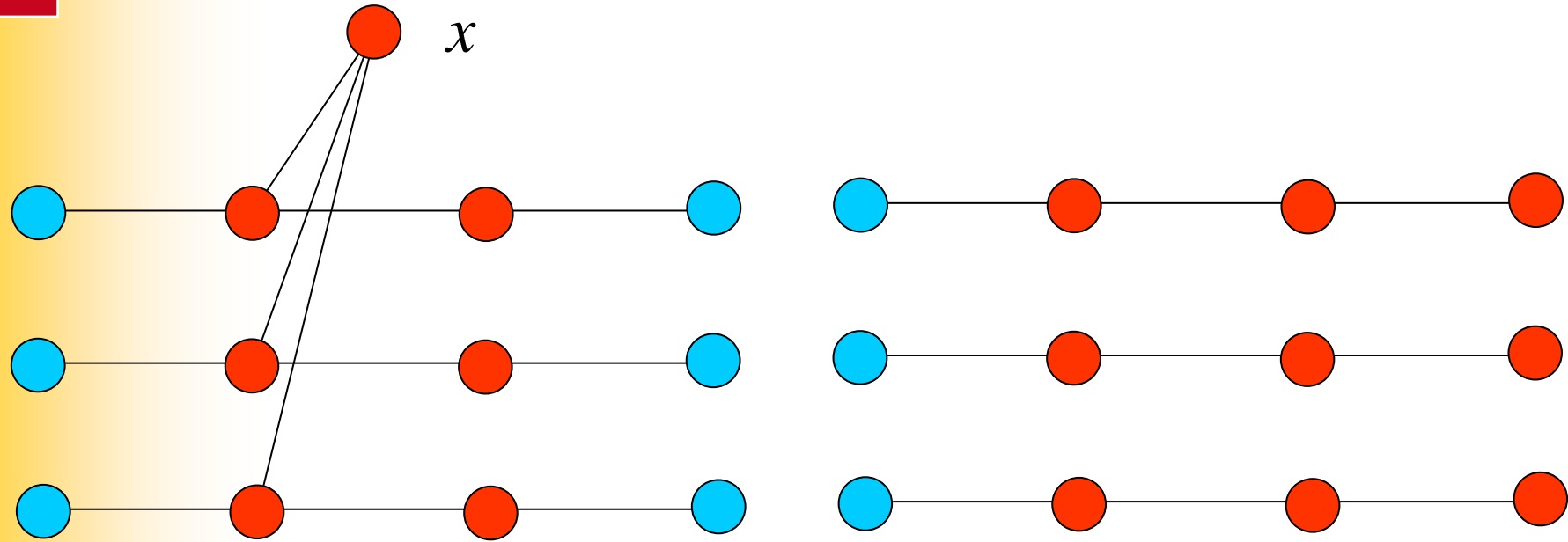
$$M_1(G_x) - M_1(G) = (n - 6)/4 \approx n/4$$

and

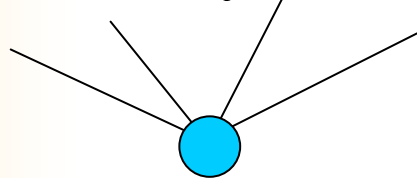
$$M_1(G_x)/M_1(G) = (3n - 6)/2n \approx 3/2.$$

- **Open problem:** can we do better?
- **Conjecture:** For any graph G , $M_1(G_x)/M_1(G) \leq 3/2$.

Proof

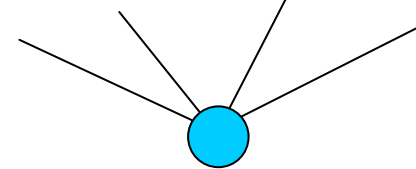


to every vertex



$$M_1(G) = 7$$

to every vertex



$$M_1(G_x) = 9 = M_1(G) + (14 - 6)/4$$

The case $r \geq 2$

Removing a vertex may involve a large decrease of M_r :

- **Theorem:** There exists a connected and r -twin-free graph G with n vertices and a vertex x of G with:

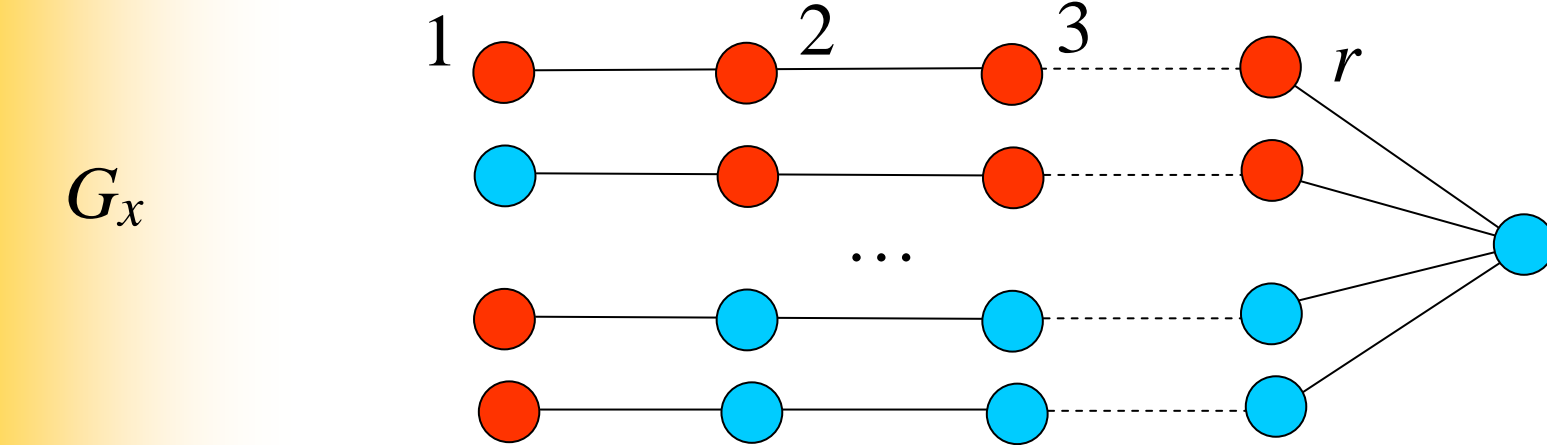
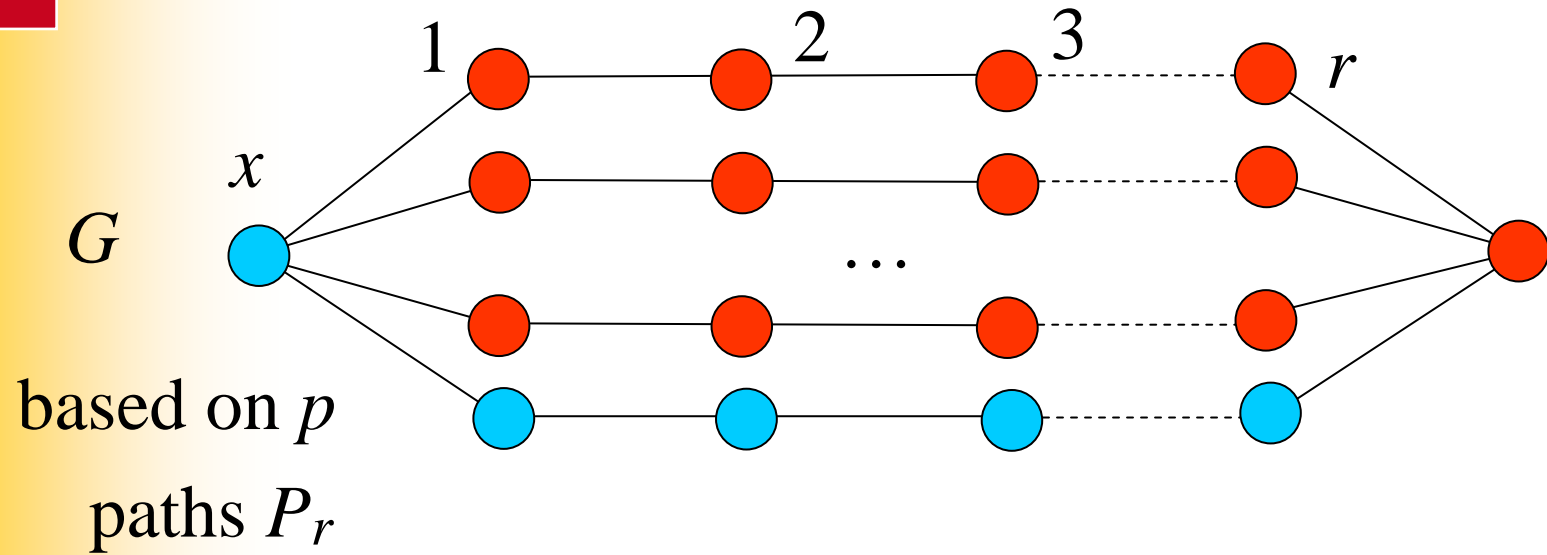
$$M_r(G) - M_r(G_x) = [(n - 3r)(r - 1) + 1]/r \approx n(r - 1)/r$$

and $M_r(G)/M_r(G_x) = r(n - r)/(n + 2r^2 - 3r - 1) \approx r$.

- **Open problem:** can we do better?
- **Conjecture:** For any graph G and any integer $r \geq 2$,

$$M_r(G)/M_r(G_x) \leq r.$$

Proof



$$M_r(G) = r(p - 1) + 1$$

$$M_r(G_x) = p + 2r - 3$$

The case $r \geq 2$, r even, G_x connected

Removing a vertex may involve a large increase of M_r :

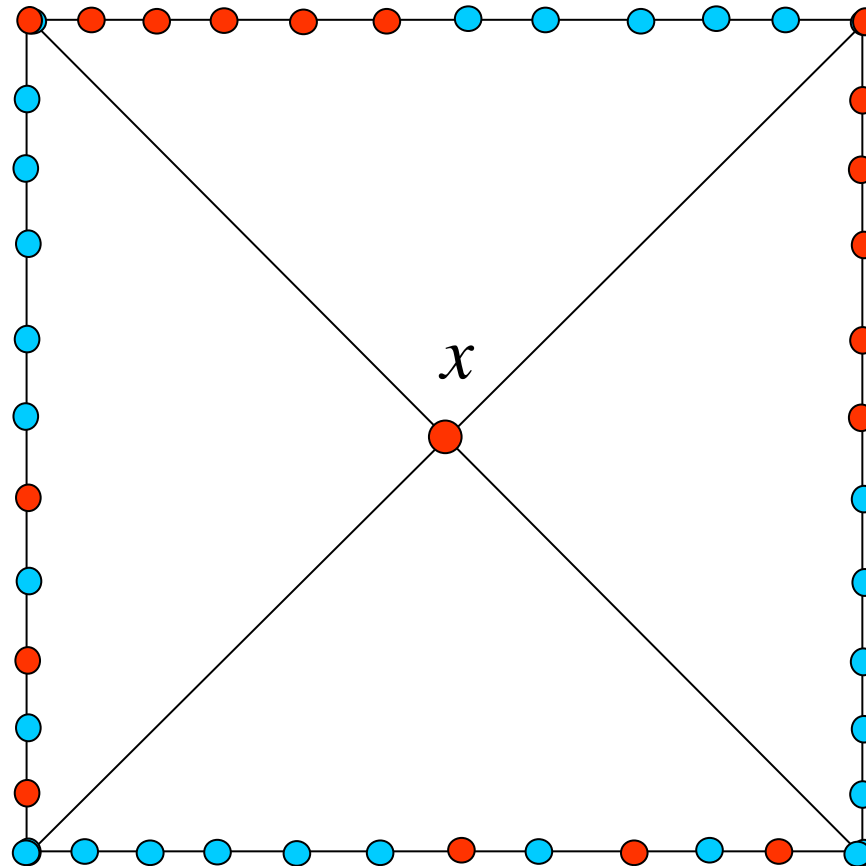
- **Theorem:** There exists a connected and r -twin-free graph G with $n + 1$ vertices and a vertex x of G with:

$$M_r(G_x) - M_r(G) \geq n/4 - (r + 1) \approx n/4$$

and $M_r(G_x)/M_r(G) = 2n/(n + 4r + 4) \approx 2$.

- **Open problem:** can we do better?
- **Conjecture:** For any connected graph G and any even integer $r \geq 2$, $M_r(G_x)/M_r(G) \leq 2$.

Example: G for $r = 6$ and $n = 48$



$$M_6(G) = 19$$

$$M_6(G_x) = M_6(C_{48}) = 24$$

The case $r \geq 3$, r odd, G_x connected

Removing a vertex may involve a large increase of M_r :

- **Theorem:** There exists a connected and r -twin-free graph G with $n + 1$ vertices and a vertex x of G with:

$$M_r(G_x) - M_r(G) \geq n(3r - 1)/12r - r \approx n/4$$

and $M_r(G_x)/M_r(G) = 6nr/[n(3r + 1) + 12r^2] \approx 6r/(3r + 1)$.

- **Open problem:** can we do better?
- **Conjecture:** For any connected graph G and any odd integer $r \geq 3$, $M_r(G_x)/M_r(G) \leq 2$.

The case $r \geq 2$, G_x disconnected

Removing a vertex may involve a larger increase of M_r :

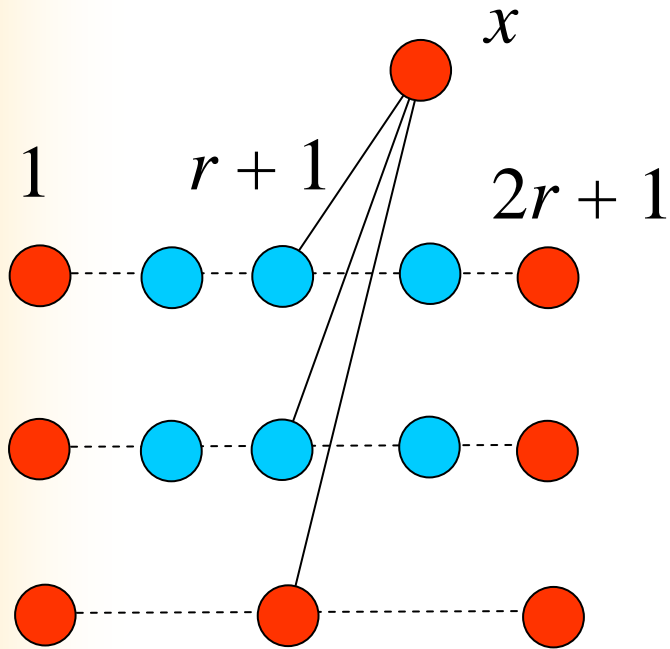
- **Theorem:** There exists a connected and r -twin-free graph G with $n + 1$ vertices and a vertex x of G with:

$$M_r(G_x) - M_r(G) \geq n(2r - 2)/(2r + 1) - 2r \approx n$$

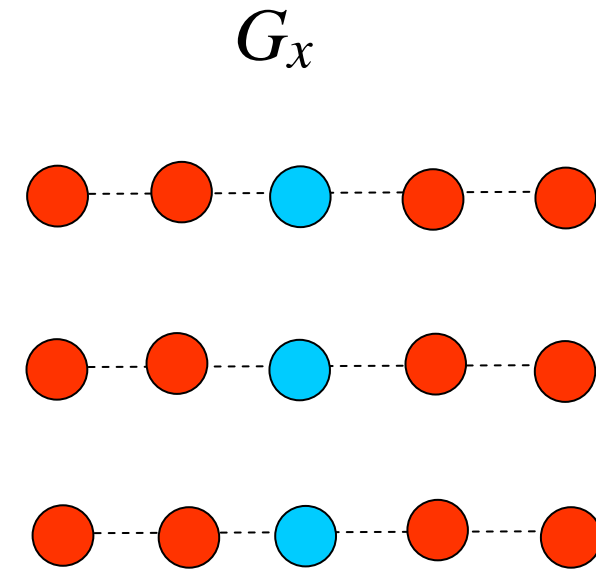
and $M_r(G_x)/M_r(G) = nr/(n + 4r^2 + 2r) \approx r$.

- **Open problem:** can we do better?
- **Conjecture:** For any graph G and any integer $r \geq 2$,
 $M_r(G_x)/M_r(G) \leq r$.

Proof: G_x is made of k copies of P_{2r+1}



$$M_r(G) = 2(r + k)$$



$$M_r(G_x) = 2kr = n - n/(2r + 1)$$

IV. Variations of $M(G)$ when an edge is added or deleted

- **Problem:** what are the maximal variations of $M_r(G)$ when an edge is added or deleted?
- More specifically, for any edge e of G , we set $G_e = G - \{e\}$. We study the quantities

$$|M_r(G) - M_r(G_e)|.$$

The case $r = 1$

Removing an edge cannot involve a large decrease or a large increase of M_1 :

- **Theorem:** Let G be any 1-twin-free graph. For any edge e of G s.t. G_e is 1-twin-free, we have:

$$M_1(G) - 2 \leq M_1(G_e) \leq M_1(G) + 2.$$

The case $r \geq 2$

Removing an edge may involve a large decrease of M_r :

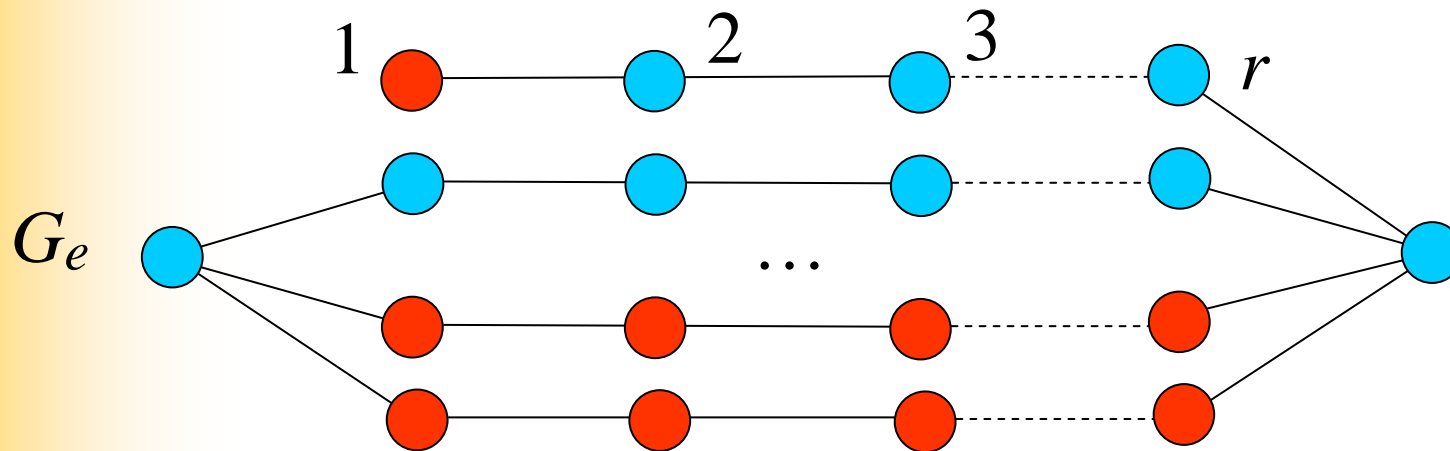
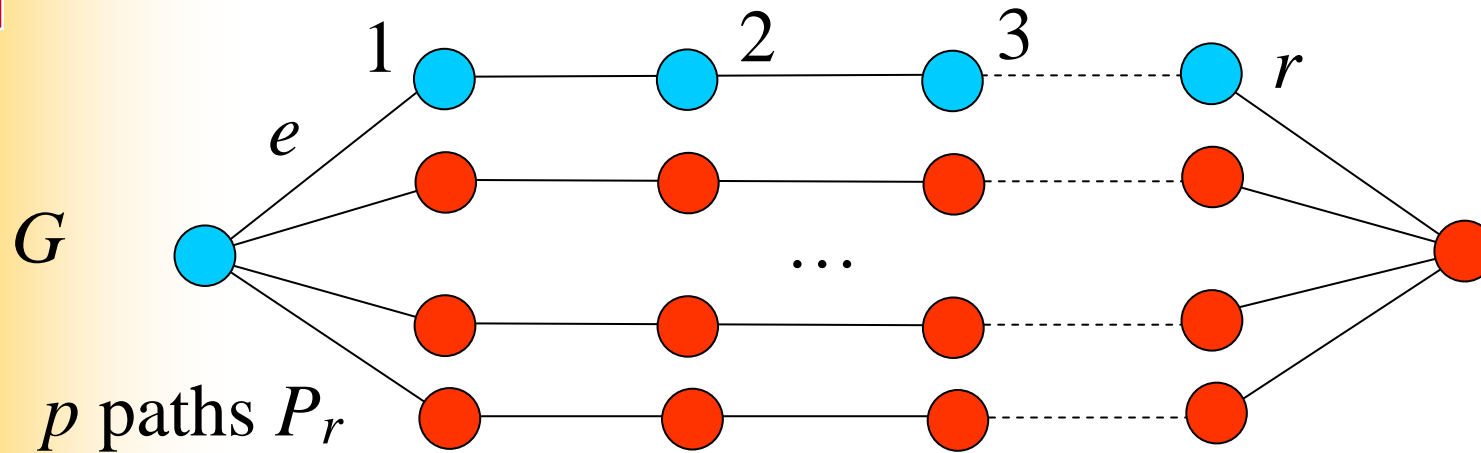
- **Theorem:** There exists a connected and r -twin-free graph G and an edge e of G with:

$$M_r(G) - M_r(G_e) = r$$

- **Open problem:** can we do better?
- **Conjecture:** For any graph G and any integer $r \geq 2$,

$$M_r(G) - M_r(G_e) \leq r.$$

Proof



$$M_r(G) = r(p - 1) + 1$$

$$M_r(G_x) = M_r(G_x) - r$$

The case $r \geq 2$, G_e connected

Removing an edge may involve a large increase of M_r :

- **Theorem:** There exists a connected and r -twin-free graph G and an edge e of G with:

$$M_r(G_e) - M_r(G) \geq r - 1.$$

- **Open problem:** can we do better?
- **Conjecture:** For any connected graph G and any even integer $r \geq 2$, $M_r(G_e) - M_r(G) \leq r$.

The case $r \geq 2$, G_e disconnected

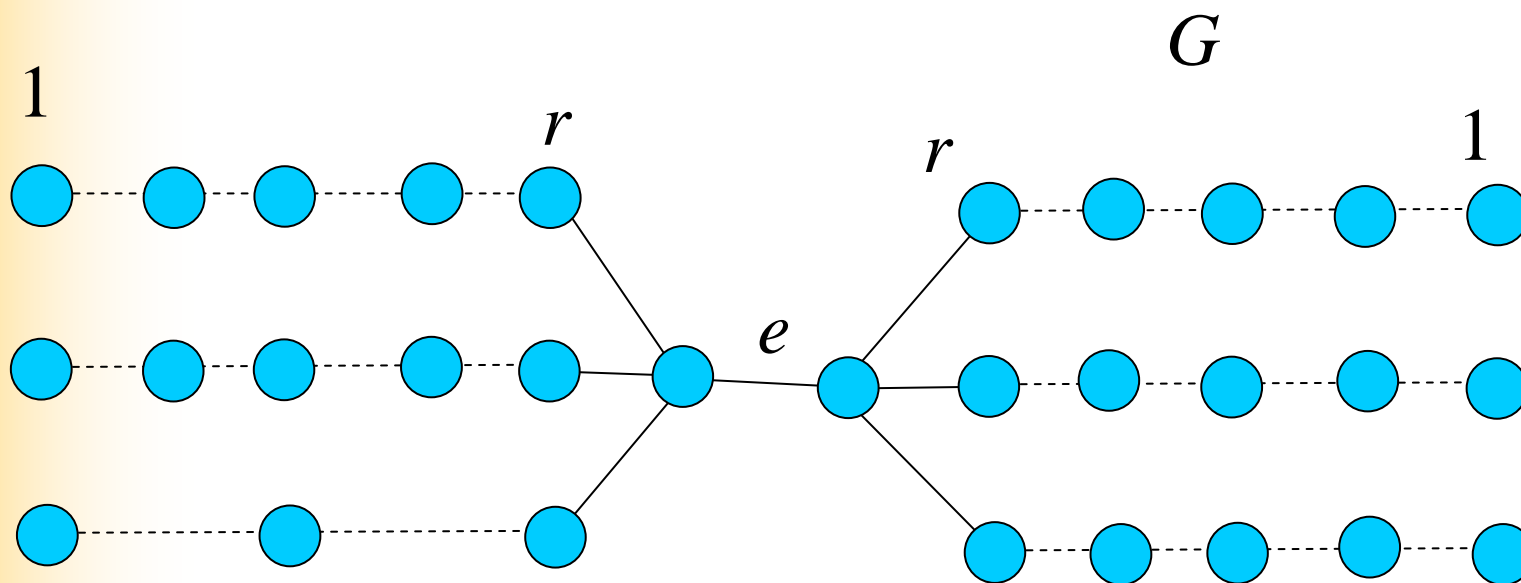
Removing a vertex may involve a larger increase of M_r :

- **Theorem:** There exists an r -twin-free graph G and an edge e of G with:

$$M_r(G_e) - M_r(G) \geq 2r - 3.$$

- **Open problem:** can we do better?
- **Conjecture:** For any graph G and any integer $r \geq 2$,
 $M_r(G_e) - M_r(G) \leq 2r.$

Proof based on the following graph:





Thank you for your attention 😊 !

References

- I. Charon, I. Honkala, O. Hudry, A. Lobstein, Structural Properties of Twin-Free Graphs, *Elec. J. of Comb.* 14 (1), R 16, 2007.
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