

Domination and Independent Domination in Direct Products of Graphs

Sandi Klavžar

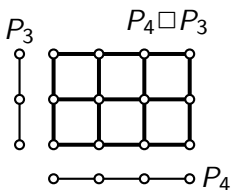
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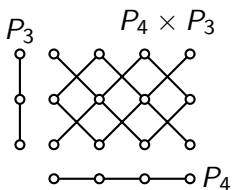
Bordeaux Workshop on Identifying Codes
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Joint work with [Boštjan Brešar](#), [Gašper Mekiš](#), [Douglas Rall](#)

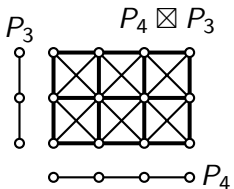
- 1 Direct product
- 2 Domination of direct products
- 3 Idomatic partitions
- 4 Four factors
- 5 Some problems



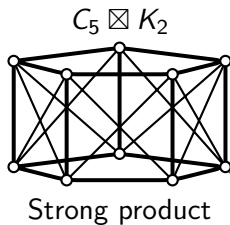
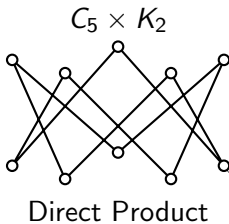
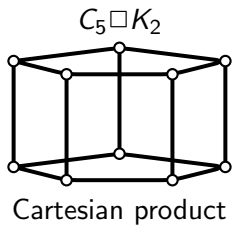
Cartesian product

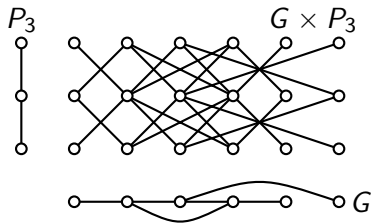


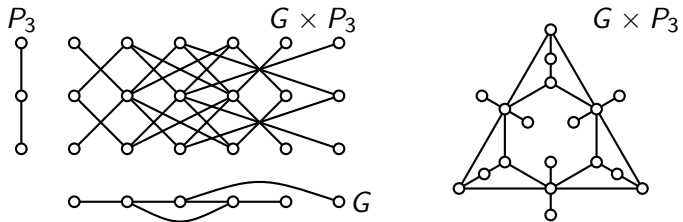
Direct product



Strong product







Infinite graphs and products

- P_∞ the two-way infinite path. Then **the square grid**:

$$P_\infty \square P_\infty .$$

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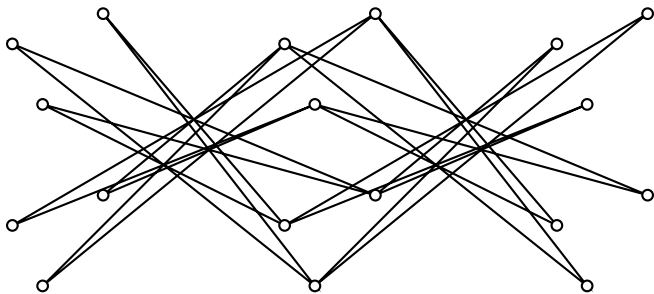
- **3D King grid**: $P_\infty^{(0)} \times P_\infty^{(0)} \times P_\infty^{(0)}$.

Basic properties

- Operation \times is associative and commutative.

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$$P_3 \times K_2 \times K_3$$

Basic properties cont'd

- $|V(G \times H)| = |V(G)| \cdot |V(H)|$.

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Proposition (Kim, 1991; Abay-Asmerom, Hammack, 2004)

The distance $d_{G \times H}((g, h), (g', h'))$ is the smallest d such that there is an g, g' -walk of length d in G and an h, h' -walk of length d in H . In particular, if such walks do not exist, then (g, h) and (g', h') are in different connected components of $G \times H$.

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Proposition

Every graph G has a prime factor decomposition with respect to \times . The number of prime factors is at most $\log_2 |G|$.

Prime factorizations cont'd

Theorem

Prime factorization with respect to the direct product is neither unique in the class of disconnected graphs with loops nor in the class of connected simple graphs.

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$$N \times \Delta = (I \times \uparrow) \times \Delta = I \times (\uparrow \times \Delta) = I \times \triangle$$

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Let G be a finite, connected, nonbipartite graph in Γ_0 . Then G has unique prime factor decomposition with respect to the direct product in Γ_0 .

Prime factorizations cont'd

Γ_0 ... class of graphs with loops

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Theorem (Imrich, 1998)

The prime factor decomposition of finite, connected, nonbipartite graphs in Γ_0 with respect to the direct product can be determined in polynomial time.

Recent progress

- G, H bipartite, when the connected components of $G \times H$ are isomorphic? Completely resolved [[Hammack, 2008](#)].

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- Connectivity.

Hedetniemi's conjecture

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Conjecture (Hedetniemi 1966)

For any graphs G and H ,

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For every pair of graphs G and H ,

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- Infinite family G_n , such that $\gamma(G_n \times G_n) \leq \frac{7}{9}\gamma(G_n)\gamma(G_n)$ [Zmazek, K., 1996]

Best upper bound

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- $|(S \times \bar{T}) \cup (\bar{S} \times T)| \leq 3\gamma(G)\gamma(H)$.

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Lower bound

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Theorem (Nowakowski, Rall, 1996)

For any graphs G and H with no isolated vertices,

$$\gamma(G \times H) \geq \max\{\rho(G)\gamma_t(H), \rho(H)\gamma_t(G)\}.$$

Products of complete graphs

Proposition (Mekiš, 2010)

If $G = \times_{i=1}^r K_{n_i}$, with $r \geq 3$ and $n_i \geq r + 1$ for all i , then $\gamma(G) = r + 1$.

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Then Proposition gives

$$\gamma(G)\gamma(G) - \gamma(G \times G) = (n + 1)(n + 1) - (2n + 1) = n^2.$$

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$$V(K_n) = [n] = \{1, 2, \dots, n\}.$$

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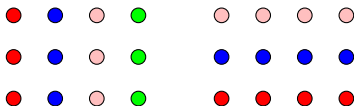
[Dunbar, Hedetniemi, Hedetniemi, Jacobs, Knisely, Laskar, Rall, 2000]

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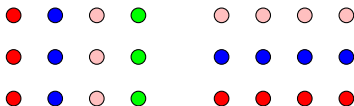
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Problem: Characterize idomatic partitions for three or more complete factors.

Three factors [Valencia-Pabon, 2010]

Lemma

I independent dominating set of $K_{n_1} \times K_{n_2} \times K_{n_3}$ containing two vertices with two equal coordinates. Then $I = pr_i^{-1}(k)$, where $i \in [3]$ and $k \in [n_i]$.

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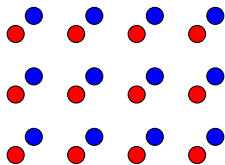
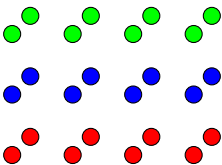
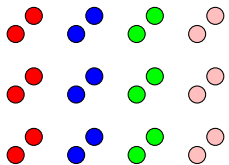
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Theorem

$K_{n_1} \times K_{n_2} \times K_{n_3}$ has an idomatic n_i -partition of Type A for each $i \in [3]$. Such partitions are the only idomatic partitions of Type A.

Three factors cont'd



Three factors cont'd

Lemma

I independent dominating set of $K_{n_1} \times K_{n_2} \times K_{n_3}$ containing no two vertices with two equal coordinates. Then

$$I = \{(\alpha_0, \alpha_1, \alpha_2), (\alpha_0, \beta_1, \beta_2), (\beta_0, \alpha_1, \beta_2), (\beta_0, \beta_1, \alpha_2)\}$$

for some $\alpha_i, \beta_i \in [n_i]$, with $\alpha_i \neq \beta_i$ and $i \in [3]$.

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- $I = \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}$

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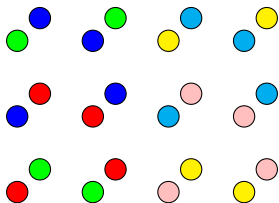
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Theorem

$K_{n_1} \times K_{n_2} \times K_{n_3}$ has an idomatic partition of Type B if and only if at least two of n_i 's are even.



Three factors cont'd

Type C partition: partition of $V(G)$ with at least one type A part and at least one type B part.

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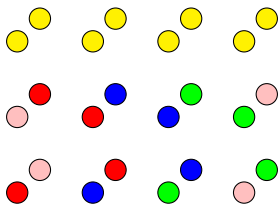
$K_{n_1} \times K_{n_2} \times K_{n_3}$ has an idomatic $(q_1 + q_2)$ -partition of Type C if and only if there exists $i \in [3]$ such that $n_i - q_1 > 1$ and $K_{n_i - q_1} \times K_{n_j} \times K_{n_k}$ admits an idomatic q_2 -partition of Type B. (Here $\{i, j, k\} = \{1, 2, 3\}$.)

Three factors cont'd

Type C partition: partition of $V(G)$ with at least one type A part and at least one type B part.

Theorem

$K_{n_1} \times K_{n_2} \times K_{n_3}$ has an idomatic $(q_1 + q_2)$ -partition of Type C if and only if there exists $i \in [3]$ such that $n_i - q_1 > 1$ and $K_{n_i - q_1} \times K_{n_j} \times K_{n_k}$ admits an idomatic q_2 -partition of Type B. (Here $\{i, j, k\} = \{1, 2, 3\}$.)



Some notation

- For $G = \times_{i=1}^k K_{n_i}$, $u = (u_1, \dots, u_k)$, and $v = (v_1, \dots, v_k)$, let

$$e(u, v) = |\{i \mid u_i = v_i\}|.$$

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- u and v are adjacent in G if and only if $e(u, v) = 0$.
- $I \subseteq V(G)$ independent if and only if $e(u, v) > 0$, $u, v \in I$.
- Let $X \subseteq V(G)$ be an independent set of $G = \times_{i=1}^k K_{n_i}$. Let

$$\{e(u, v) \mid u, v \in X, u \neq v\} = \{j_1, \dots, j_r\}.$$

Then we say that X is a T_{j_1, \dots, j_r} -set.

Possibilities

Let $I \subseteq V(G)$, $G = \times_{i=1}^t K_{n_i}$, be an independent and dominating set of G .

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Four factors ($t = 3$)

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- (iii) T_3 -set,
- (iv) $T_{1,2}$ -set,
- (v) $T_{1,3}$ -set,
- (vi) $T_{2,3}$ -set,
- (vii) $T_{1,2,3}$ -set.

T_2 -sets and $T_{2,3}$ -set are not admissible

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- $e((0, 0, 1, 1), (1, 0, 0, 0)) = 1 \Rightarrow (1, 0, 0, 0) \notin I$.
- There exists $(a, b, c, d) \in I$ such that (a, b, c, d) dominates $(1, 0, 0, 0)$.
- In particular, $b, c, d \neq 0$.
- But then $e((a, b, c, d), (0, 0, 0, 0)) \leq 1$, a contradiction.

$T_{1,2}$ -partitions

Let $G = K_2 \times K_2 \times K_2 \times K_4$. Then the sets

$$I_1 = \{(0, 0, 0, 0), (1, 1, 1, 0), (0, 1, 1, 1), (1, 0, 0, 1), \\ (1, 0, 1, 2), (0, 1, 0, 2), (1, 1, 0, 3), (0, 0, 1, 3)\},$$

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and

$$I_2 = \{(u_1 + 1 \bmod 2, u_2, u_3, u_4) \mid (u_1, u_2, u_3, u_4) \in I_1\},$$

$$I_3 = \{(u_1, u_2 + 1 \bmod 2, u_3, u_4) \mid (u_1, u_2, u_3, u_4) \in I_1\},$$

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form an idomatic partition of G .

$T_{1,2,3}$ -partitions

- For $K_{n_1} \times K_{n_2} \times K_{n_3} \times K_{n_4}$,

$$I = [n_0] \times [n_1] \times [n_2] \times \{i\},$$

is a dominating $T_{1,2,3}$ -set.

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- For $\times_{i=1}^t K_{n_i}$,

$$I_j = [n_1] \times \cdots \times [n_{\ell-1}] \times \{j\} \times [n_{\ell+1}] \times \cdots \times [n_t]$$

is a dominating $T_{1,2,\dots,t-1}$ -set, where $j \in [n_\ell]$.

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- Let $n_\ell = \min\{n_i \mid 1 \leq i \leq k\}$. Then (as Hedetniemi's conjecture holds for complete graphs) $\chi(G) = n_\ell$ and therefore $\chi(G) = \chi_f(G)$.

$T_{1,2,3}$ -partitions cont'd

For $K_2 \times K_2 \times K_2 \times K_2$,

$$I_1 = \{(0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), \\ (0, 0, 0, 1), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1)\}$$

and

$$I_2 = \{(1, 1, 1, 1), (0, 1, 1, 1), (1, 0, 1, 1), (1, 1, 0, 1), \\ (1, 1, 1, 0), (0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0)\}$$

are both dominating $T_{1,2,3}$ -sets.

Characterization of T_1 -sets

Theorem

Let I be a T_1 -set of $K_{n_1} \times K_{n_2} \times K_{n_3} \times K_{n_4}$. Then I is a dominating set if and only if $n_i \geq 3$ and

$$I = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_3), (\alpha_0, \beta_1, \beta_2, \beta_3), (\alpha_0, \gamma_1, \gamma_2, \gamma_3), \\ (\beta_0, \alpha_1, \beta_2, \gamma_3), (\beta_0, \gamma_1, \alpha_2, \beta_3), (\beta_0, \beta_1, \gamma_2, \alpha_3), \\ (\gamma_0, \alpha_1, \gamma_2, \beta_3), (\gamma_0, \beta_1, \alpha_2, \gamma_3), (\gamma_0, \gamma_1, \beta_2, \alpha_3)\},$$

where $\alpha_i, \beta_i, \gamma_i$ are pairwise different.

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$$I = \{(0, 0, 0, 0), (0, 1, 1, 1), (0, 2, 2, 2), \\ (1, 0, 1, 2), (1, 2, 0, 1), (1, 1, 2, 0), \\ (2, 0, 2, 1), (2, 1, 0, 2), (2, 2, 1, 0)\}.$$

Characterization of T_1 -partitions

Theorem

$K_{n_1} \times K_{n_2} \times K_{n_3} \times K_{n_4}$ admits an idomatic partition into T_1 -sets if and only if $n_i \geq 3$ and there exist indices $j, k \in [4]$, $j \neq k$, such that $3|n_j$ and $3|n_k$.

More than 4 factors

Theorem (Gravier, Mekiš, Mollard, 2010)

Let $G = \times_{i=1}^t K_{n_i}$ with $t \geq 3$ and $n_i \geq t - 1$ for all i and let $0, 1, a_2, \dots, a_{t-2}$ be pairwise different (all) elements from $GF[p^k]$. If $t - 1 = p^k$ for some prime number p and some positive integer k , then the set

$$I = \mathcal{L}\{(0, 1, 1, \dots, 1), (1, 0, 1, a_2, \dots, a_{t-2})\},$$

the two dimensional subspace of $\times_{i=1}^t GF[p^k]$, is a dominating T_1 -set.

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To be hopefully continued ...

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Characterize $T_{1,2}$ -sets and $T_{1,2,3}$ -sets in direct products of four complete graphs.

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Gravier, Moncel, Semri, Identifying codes of Cartesian product of two cliques of the same size, Electron. J. Combin. 15 (2008).

Problem

Identifying codes of the direct product of two (or more) complete graphs (of the same size).

Thank you.