

On k -domination and j -dependence in graphs

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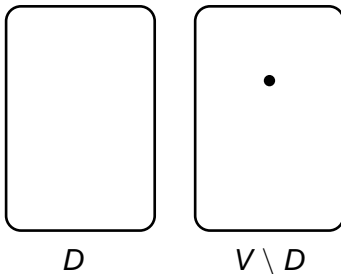
joint work with **Ryan Pepper**
University of Houston-Downtown

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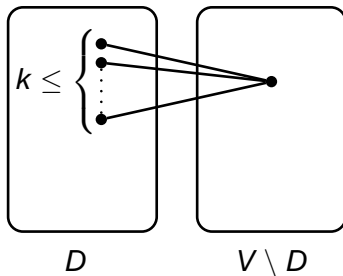
Definitions

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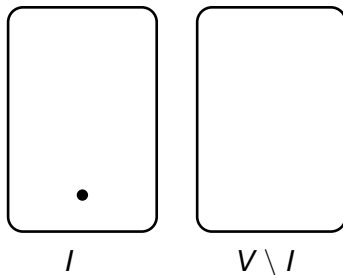
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- If D is a minimum k -dominating set, then $\gamma_k(G) = |D|$ is the *k -domination number* of G .

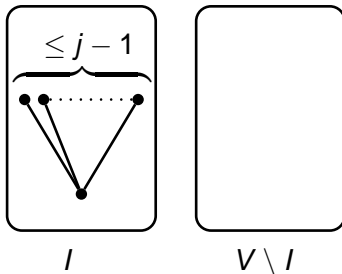
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- $\gamma_1(G) = \gamma(G)$

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- $\alpha_1(G) = \alpha(G)$, 1-dependent = independent.

k-domination and *j*-dependence

Favaron, 1985

$$\gamma_k(\mathbf{G}) \leq \alpha_k(\mathbf{G})$$

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Jacobson, Peters, Rall, 1990

$$\gamma_{\delta-j+1}(\mathbf{G}) + \alpha_j(\mathbf{G}) \leq n$$

Favaron, 2005

$$\gamma_k(\mathbf{G}) + \alpha_{\Delta-k+1}(\mathbf{G}) \geq n$$

$$k, j, m \geq 1 \text{ integers s. t. } m = k + j - 1$$

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Pepper, 2010

$$\gamma_k(G) + \alpha_j(H_m) \leq n$$

H. & P.

$$\gamma_k(G_m) + \alpha_j(G) \geq n(G_m)$$

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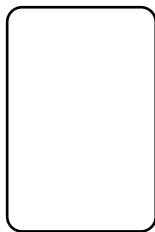
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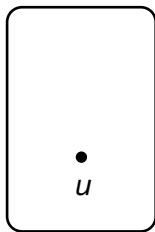
$I = \text{max.}$
 j -dep. set of H_m



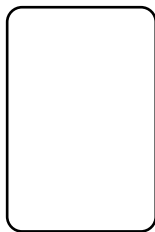
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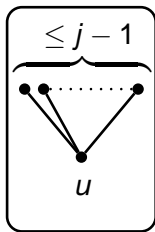
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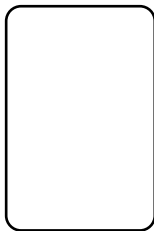
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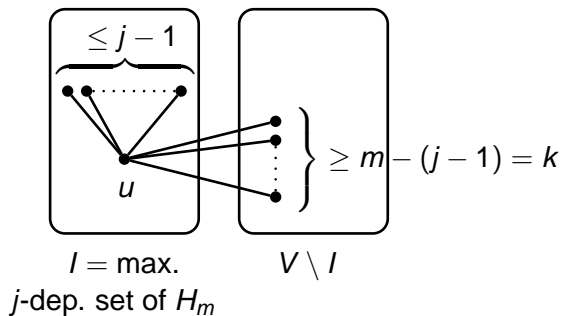
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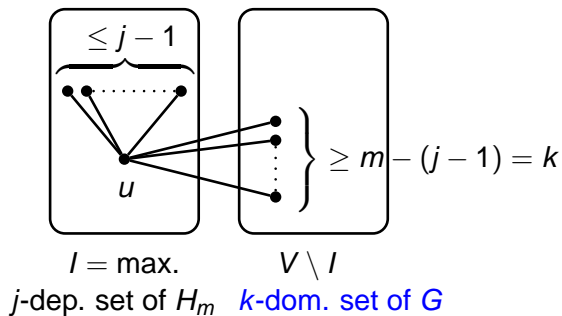
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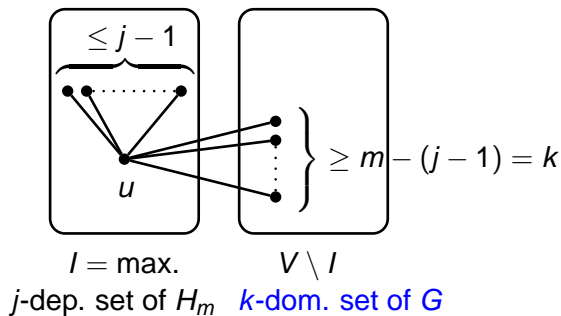
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$$\Rightarrow \gamma_k(G) \leq n - \alpha_j(H_m)$$

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$D = \text{min.}$

k -dom. set of G_m



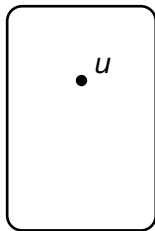
$V(G_m) \setminus D$

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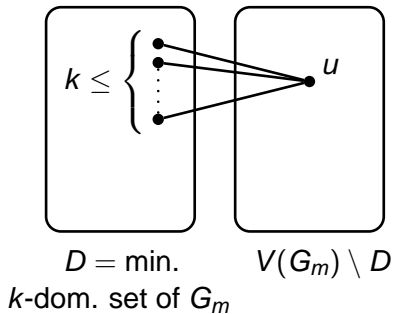
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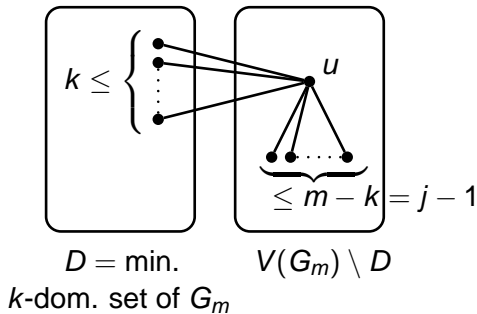
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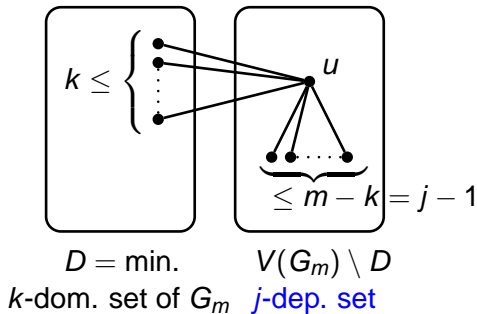
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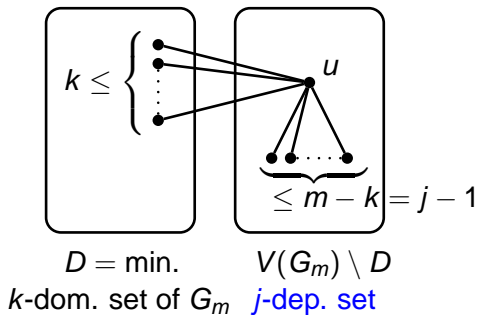
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$$\Rightarrow \alpha_j(G) \geq n(G_m) - \gamma_k(G_m)$$

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Pepper, 2010

$$\gamma_k(G) + \alpha_j(H_m) \leq n$$

H. & P.

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$$m = \delta \Rightarrow H_m = G$$

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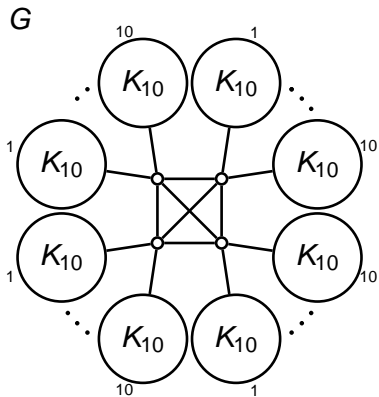
Jacobson, Peters, Rall, 1990

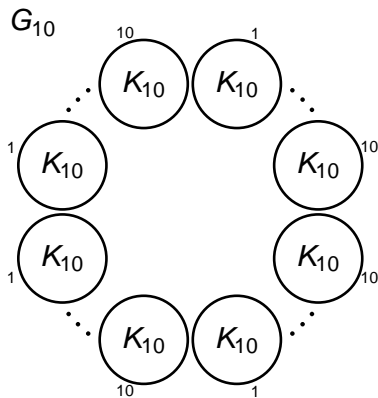
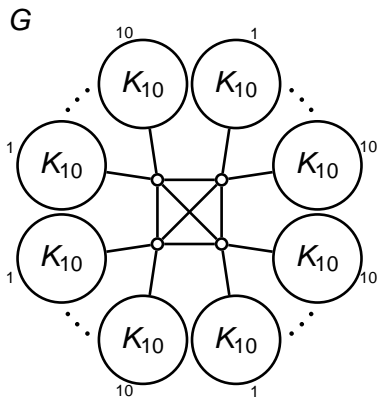
$$\gamma_k(G) + \alpha_j(G) \leq n$$

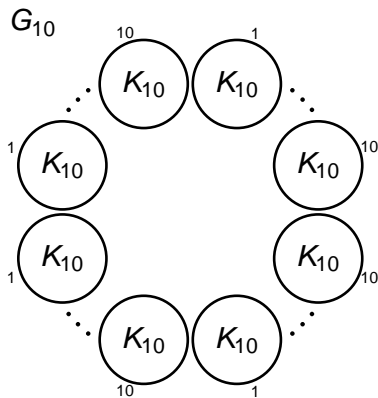
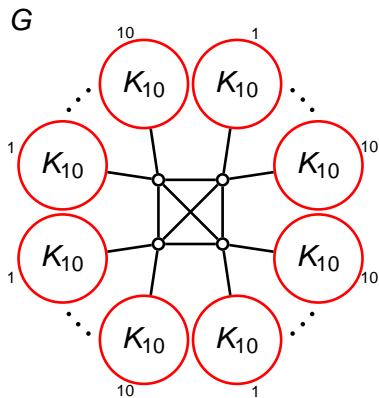
Favaron, 2005

$$\gamma_k(G) + \alpha_j(G) \geq n$$

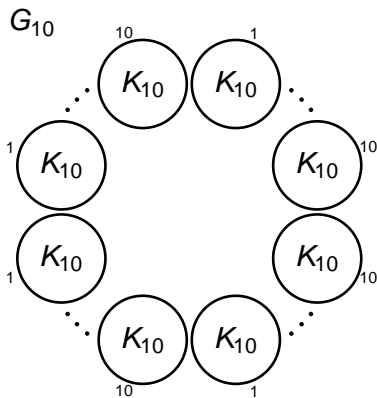
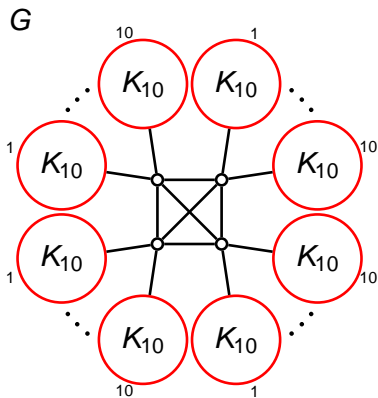
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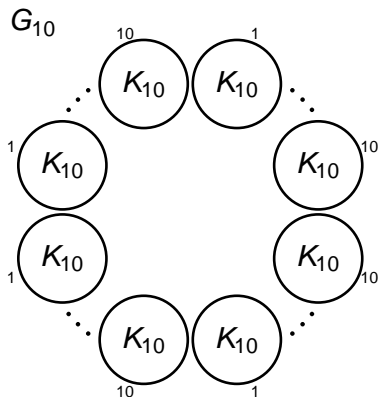
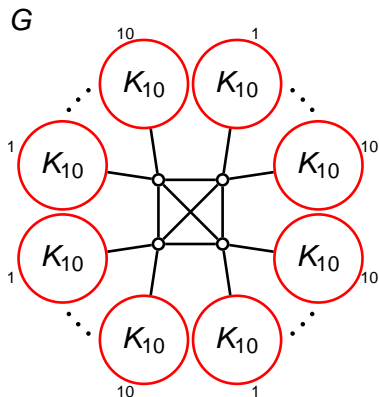
Example:

$$\alpha_{10}(G) = 400$$

Example:

$$\alpha_{10}(G) = \gamma_{94}(G) = 400$$

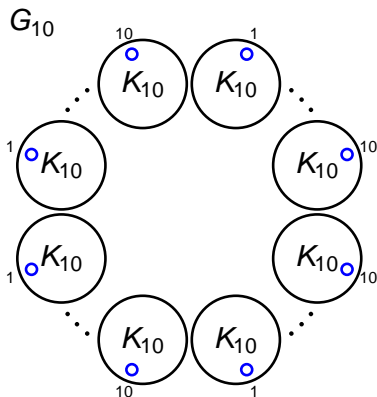
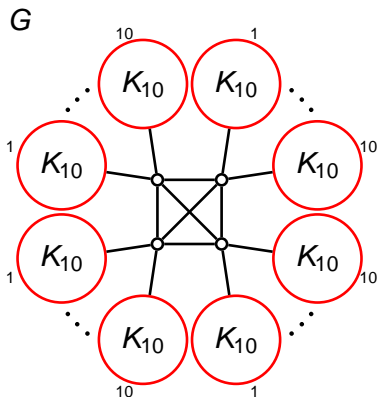
Example:



$$\alpha_{10}(G) = \gamma_{94}(G) = 400$$

$$\begin{aligned} \alpha_{10}(G) &\geq n - \gamma_{94}(G) \\ &= 404 - 400 = 4 \end{aligned}$$

Example:



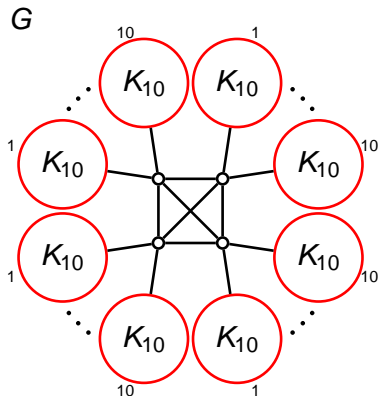
$$\alpha_{10}(G) = \gamma_{94}(G) = 400$$

$$\alpha_{10}(G) \geq n - \gamma_{94}(G)$$

$$= 404 - 400 = 4$$

$$\gamma(G_{10}) = 40$$

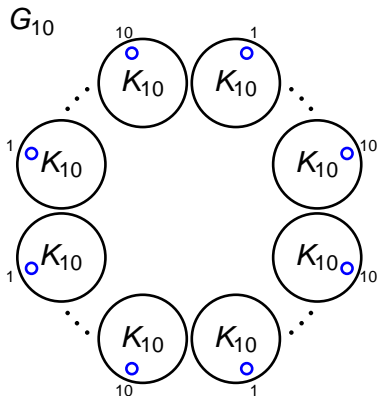
Example:



$$\alpha_{10}(G) = \gamma_{94}(G) = 400$$

$$\alpha_{10}(G) \geq n - \gamma_{94}(G)$$

$$= 404 - 400 = 4$$



$$\gamma(G_{10}) = 40$$

$$\alpha_{10}(G) \geq n(G_{10}) - \gamma(G_{10})$$

$$= 400 - 40 = 360$$

$$\delta = k + j - 1$$

Jacobson, Peters, Rall, 1990

$$\gamma_k(\mathbf{G}) + \alpha_j(\mathbf{G}) \leq n$$

$$\Delta = k + j - 1$$

Favaron, 2005

$$\gamma_k(\mathbf{G}) + \alpha_j(\mathbf{G}) \geq n$$

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Favaron, 2005

$$\gamma_k(\mathbf{G}) + \alpha_j(\mathbf{G}) \geq n$$

$$\mathbf{G} \text{ } r\text{-regular} \Rightarrow \gamma_k(\mathbf{G}) + \alpha_j(\mathbf{G}) = n$$

$$\text{for all } j, k \geq 1 \text{ s.t. } r = k + j - 1$$

$$\delta = k + j - 1$$

Jacobson, Peters, Rall, 1990

$$\gamma_k(\mathbf{G}) + \alpha_j(\mathbf{G}) \leq n$$

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Favaron, 2005

$$\gamma_k(\mathbf{G}) + \alpha_j(\mathbf{G}) \geq n$$

\mathbf{G} r -regular $\Rightarrow \gamma_k(\mathbf{G}) + \alpha_j(\mathbf{G}) = n$
for all $j, k \geq 1$ s.t. $r = k + j - 1$

When is regularity necessary?

$\gamma_k(\mathbf{G}) + \alpha_j(\mathbf{G}) = n$ and $\gamma_{k'}(\mathbf{G}) + \alpha_{j'}(\mathbf{G}) = n$
for every pair of integers k, j and k', j' such that
 $k + j - 1 = \delta$ and $k' + j' - 1 = \Delta$

\Leftrightarrow

\mathbf{G} is regular

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$\gamma(\mathbf{G}) + \alpha_\delta(\mathbf{G}) = n$ and $\gamma_{\Delta-\delta+1}(\mathbf{G}) + \alpha_\delta(\mathbf{G}) = n$

\Leftrightarrow

\mathbf{G} is either regular

or $\gamma(\mathbf{G}) = \gamma_2(\mathbf{G})$ and \mathbf{G} is semiregular

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\Leftrightarrow

\mathbf{G} is either regular

or $\gamma(\mathbf{G}) = \gamma_2(\mathbf{G})$ and \mathbf{G} is semiregular

If \mathbf{G} C_4 -free, then:

$\gamma(\mathbf{G}) + \alpha_\delta(\mathbf{G}) = n$ and $\gamma_{\Delta-\delta+1}(\mathbf{G}) + \alpha_\delta(\mathbf{G}) = n$

\Leftrightarrow

\mathbf{G} is regular

Some implications

$$m = k + j - 1$$

$$H_m = G[\{v \in V \mid \deg(v) \geq m\}], G_m = G[\{v \in V \mid \deg(v) \leq m\}]$$

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$$\gamma_k(G) + \alpha_j(H_m) \leq n$$

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Examples for $j = 1$ and $k = \delta \Rightarrow H_m = G$:

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Fajtlowicz, 1977

$$G \text{ } K_q\text{-free} \Rightarrow \alpha(G) \geq \frac{2}{\Delta+q}n$$

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$$G \text{ planar and } K_3\text{-free} \Rightarrow \alpha(G) \geq \frac{3}{8}n$$

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Examples for $j = 1$ and $k = \delta \Rightarrow H_m = G$:

Staton, 1979

$$G \text{ } K_3\text{-free, } \Delta \geq 3 \Rightarrow \alpha(G) \geq \frac{5}{5\Delta-1}n$$

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$$G \text{ cubic, } K_3\text{-free} \Rightarrow \gamma_3(G) \leq \frac{9}{14}n$$

Other symmetrical results

Caro & Roditty, 1990

$$\delta \geq k \frac{r}{r-1} - 1$$

$$\Rightarrow \gamma_k(\mathbf{G}) \leq \frac{r-1}{r} n$$

H. & P.

$$\Delta \leq jr - 1$$

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Favaron, H., Volkmann, 2008

$$\delta \geq k$$

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Hopkings & Staton, 1986

$$\alpha_j(\mathbf{G}) \geq \frac{n}{\lfloor \frac{\Delta}{j} \rfloor + 1}$$

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Fink & Jacobson, 1985

$$\gamma_k(\mathbf{G}) \geq \frac{k}{\Delta+k} n$$

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$$\alpha_j(\mathbf{G}) \leq \frac{\Delta}{\Delta + \delta - j + 1} n$$

Fink & Jacobson, 1985

$$\gamma_k(\mathbf{G}) \geq n - \frac{e(\mathbf{G})}{k}$$

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$$\alpha_j(\mathbf{G}) \leq \frac{e(\mathbf{G})}{\delta - j + 1}$$

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$$\alpha_j(\mathbf{G}) \leq \frac{e(\mathbf{G})}{\delta - j + 1}$$

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$$\gamma_k(\mathbf{G}) \geq \frac{2e(\mathbf{G}) - (\Delta - k)n}{2\Delta + k - 1}$$

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$$\alpha_j(\mathbf{G}) \leq \frac{(3\Delta - 1)n - 2e(\mathbf{G})}{2\Delta + \delta - j}$$

MERCI BEAUCOUP!