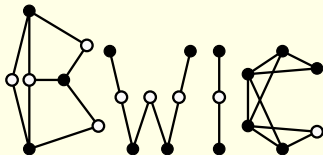


# Tolerant identifying codes using Euclidean balls

Aline Parreau

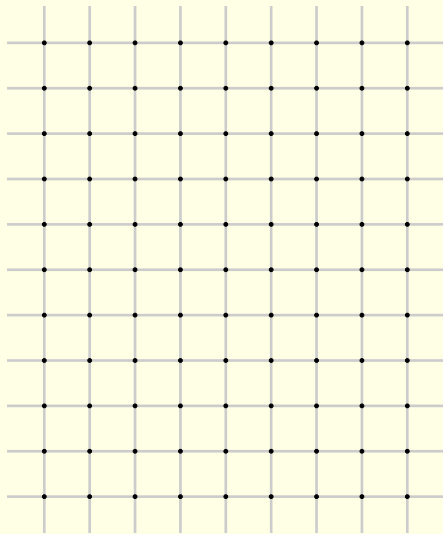
With Ville Junnila and Tero Laihonen



Bordeaux Workshop on Identifying Codes  
November 21-25, 2011

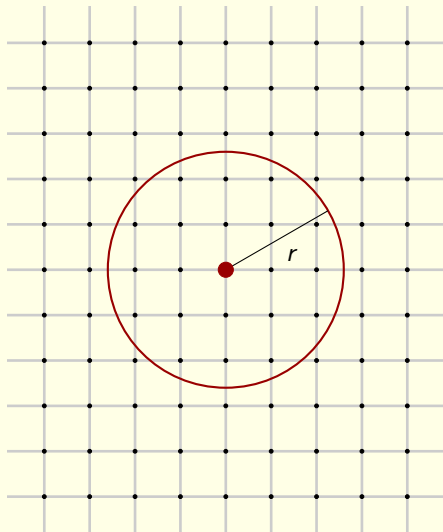
# Framework

- Vertices (sensors):  $\mathbb{Z}^2$



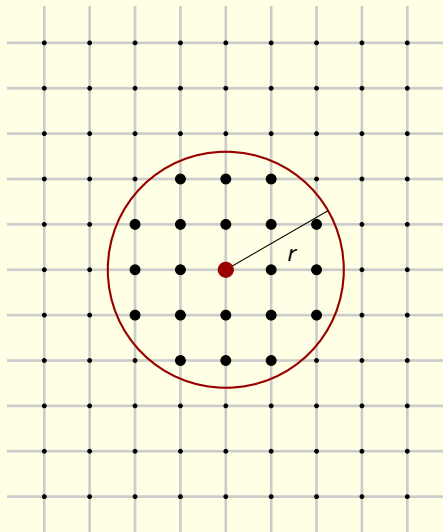
# Framework

- Vertices (sensors):  $\mathbb{Z}^2$
- Can detect faults at Euclidean distance  $r$



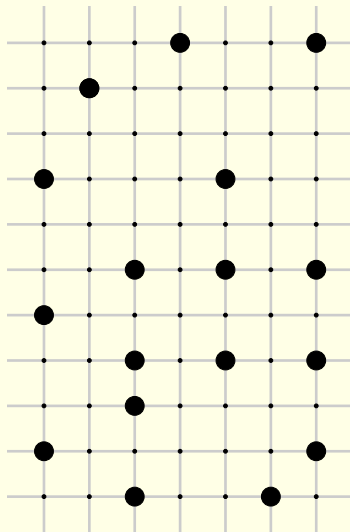
# Framework

- Vertices (sensors):  $\mathbb{Z}^2$
  - Can detect faults at Euclidean distance  $r$
- Euclidean grid



# Identifying codes in Euclidean grids

Identifying code : subset  $C \subseteq \mathbb{Z}^2$  such that:

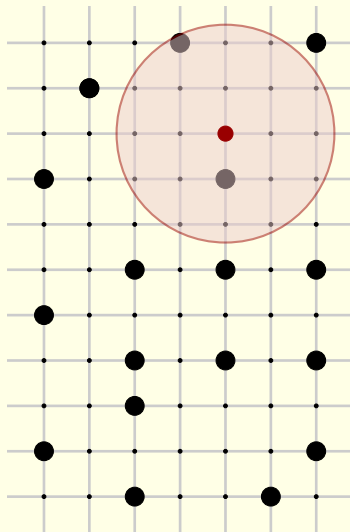


# Identifying codes in Euclidean grids

Identifying code : subset  $C \subseteq \mathbb{Z}^2$  such that:

- $C$  is an  $r$ -dominating set: for all  $u \in \mathbb{Z}^2$ ,

$$C \cap B_r(u) \neq \emptyset$$



# Identifying codes in Euclidean grids

Identifying code : subset  $C \subseteq \mathbb{Z}^2$  such that:

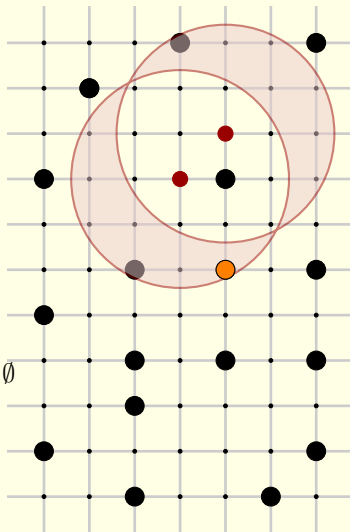
- $C$  is an  $r$ -dominating set: for all  $u \in \mathbb{Z}^2$ ,

$$C \cap B_r(u) \neq \emptyset$$

- $C$  is an  $r$ -separating set : for all  $u, v \in \mathbb{Z}^2$ ,

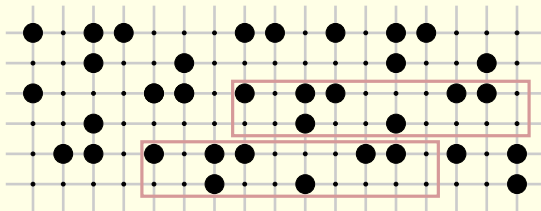
$$C \cap ((B_r(u) \setminus B_r(v)) \cup (B_r(v) \setminus B_r(u))) \neq \emptyset$$

Goal: minimize density  $D(r)$  of  $C$



# Square grid

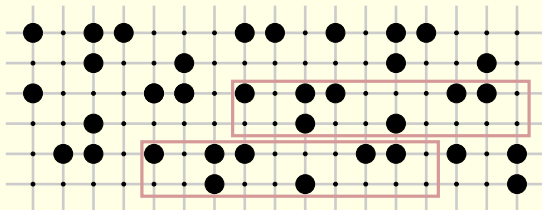
$r = 1 \rightarrow$  square grid





# Square grid

$r = 1 \rightarrow$  square grid



**Proposition** Cohen *et al*, 1999 and Ben-Haim & Litsyn, 2005

The optimal density of an identifying code in the square grid is  $D(1) = \frac{7}{20}$

## Other results for small values of $r$

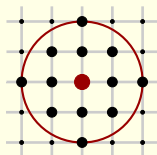
$r = \sqrt{2}$ , King grid:

$$D(\sqrt{2}) = \frac{2}{9}$$



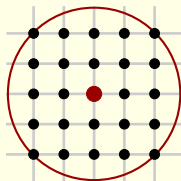
$r = 2$ , Square grid of radius 2:

$$\frac{6}{35} \leq D(2) \leq \frac{5}{29}$$



$r = 2\sqrt{2}$  King grid of radius 2:

$$D(2\sqrt{2}) = \frac{1}{8}$$



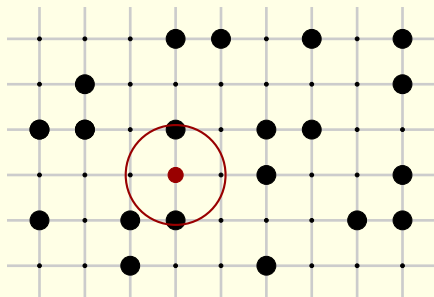
For a general radius

**Proposition** Junnila and Laihonon, 2011

General bounds:

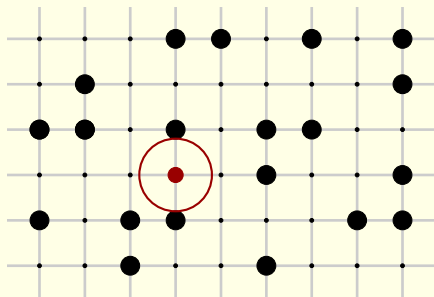
$$\frac{1}{3.22r + 4} \leq D(r) \leq \frac{1}{2\lfloor r \rfloor}$$

## Optimal codes are not robust



The code is not any more identifying if the radius changes a little.

## Optimal codes are not robust



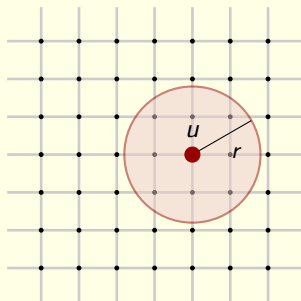
The code is not any more identifying if the radius changes a little.

# Tolerant identifying codes

Balls are between balls of radius  $r$  and  $r + \Delta$ .

$C \subseteq \mathbb{Z}^2$  is an  $(r, \Delta)$ -identifying code if  $C$  is:

- $r$ -dominating :  $C \cap B_r(u) \neq \emptyset$



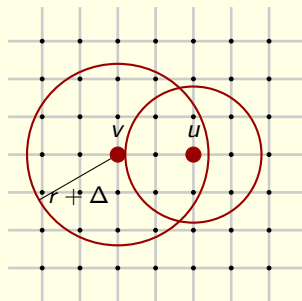
$B_r(u)$

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 $B_r(u) \setminus B_{r+\Delta}(v) \cup B_r(v) \setminus B_{r+\Delta}(u)$   
is non-empty.



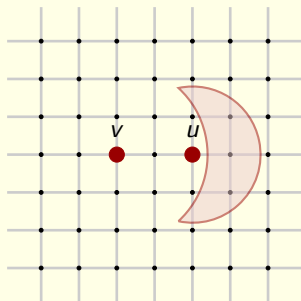
$B_r(u)$   $B_{r+\Delta}(v)$

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$B_r(u) \setminus B_{r+\Delta}(v)$



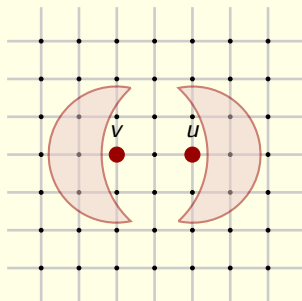
# Tolerant identifying codes

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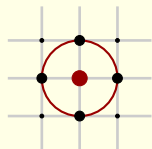
- $r$ -dominating :  $C \cap B_r(u) \neq \emptyset$
- $(r, \Delta)$ -separating:  
 $S_{r,\Delta}(u, v) =$   
 $B_r(u) \setminus B_{r+\Delta}(v) \cup B_r(v) \setminus B_{r+\Delta}(u)$   
is non-empty.

$D(r, \Delta)$ : minimal density

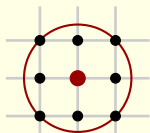


$$B_r(u) \setminus B_{r+\Delta}(v) \cup B_r(v) \setminus B_{r+\Delta}(u)$$

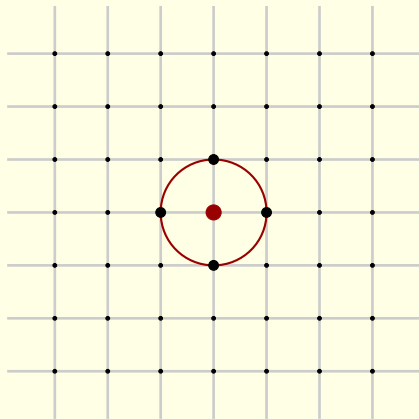
First example:  $(r, \Delta) = (1, \sqrt{2} - 1)$



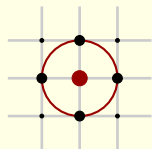
$$r = 1$$



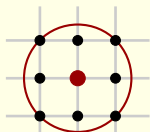
$$r + \Delta = \sqrt{2}$$



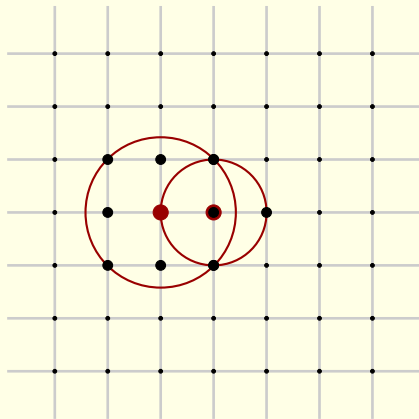
First example:  $(r, \Delta) = (1, \sqrt{2} - 1)$



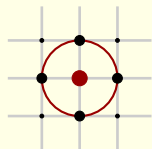
$$r = 1$$



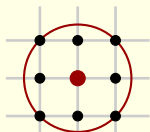
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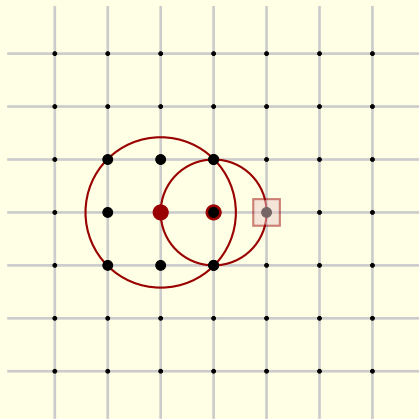
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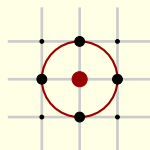
$$r = 1$$



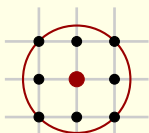
$$r + \Delta = \sqrt{2}$$



First example:  $(r, \Delta) = (1, \sqrt{2} - 1)$

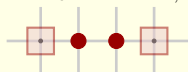


$$r = 1$$

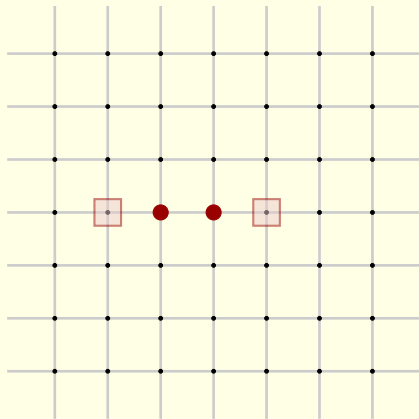


$$r + \Delta = \sqrt{2}$$

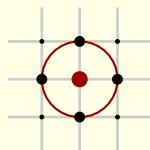
Horizontal pattern  $\mathcal{S}_{r, \Delta}(\bullet \bullet)$ :



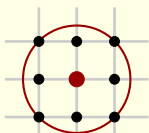
$$\Rightarrow D(r, \Delta) \geq 1/2$$



First example:  $(r, \Delta) = (1, \sqrt{2} - 1)$

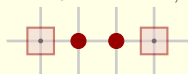


$$r = 1$$

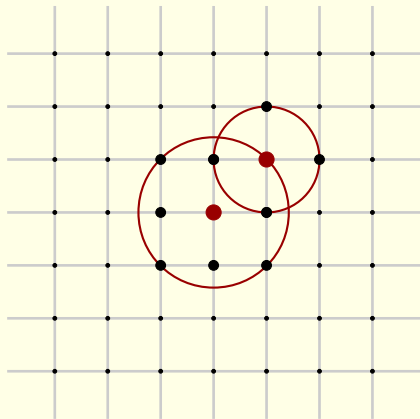


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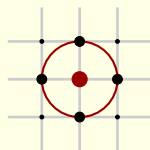
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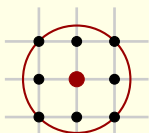
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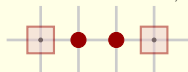


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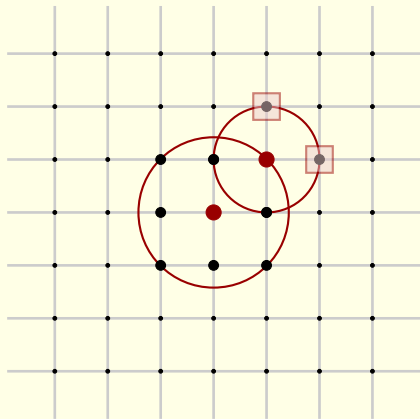


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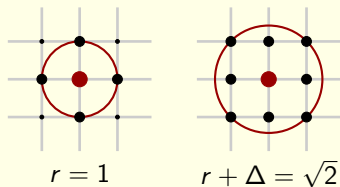
Horizontal pattern  $\mathcal{S}_{r,\Delta}(\bullet, \bullet)$ :



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First example:  $(r, \Delta) = (1, \sqrt{2} - 1)$

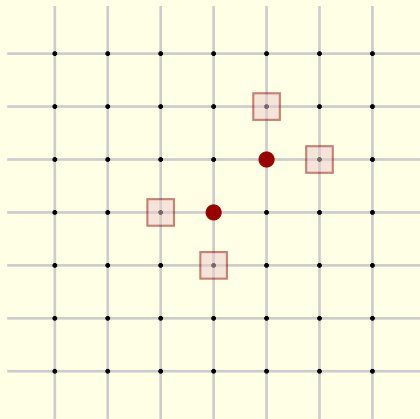
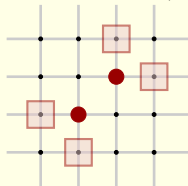


Horizontal pattern  $\mathcal{S}_{r,\Delta}(\bullet, \bullet)$ :



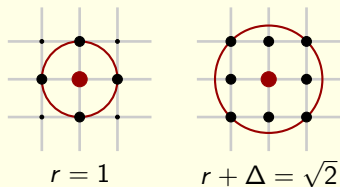
$$\Rightarrow D(r, \Delta) \geq 1/2$$

Diagonal pattern  $\mathcal{S}_{r,\Delta}(\bullet, \bullet)$ :

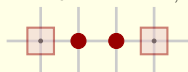




First example:  $(r, \Delta) = (1, \sqrt{2} - 1)$

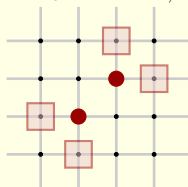


Horizontal pattern  $\mathcal{S}_{r,\Delta}(\bullet, \bullet)$ :

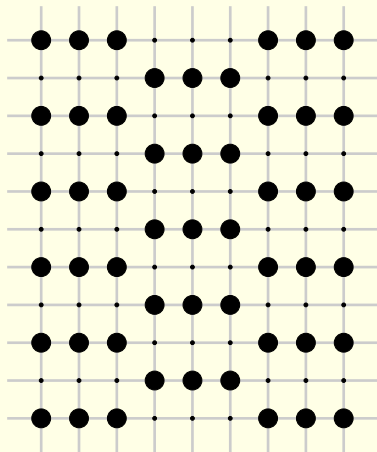


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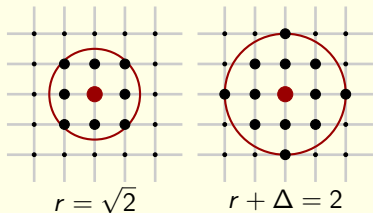
Diagonal pattern  $\mathcal{S}_{r,\Delta}(\bullet, \bullet)$ :



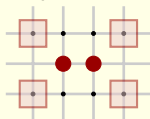
$$D(r, \Delta) = \frac{1}{2}$$



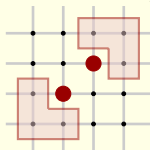
Next one:  $(r, \Delta) = (\sqrt{2}, 2 - \sqrt{2})$



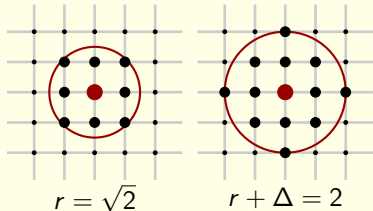
Horizontal pattern  $\mathcal{S}_{r,\Delta}(\bullet \bullet)$ :



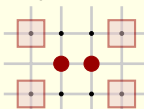
Diagonal pattern  $\mathcal{S}_{r,\Delta}(\bullet \bullet)$ :



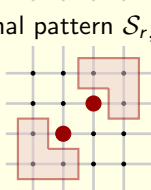
Next one:  $(r, \Delta) = (\sqrt{2}, 2 - \sqrt{2})$



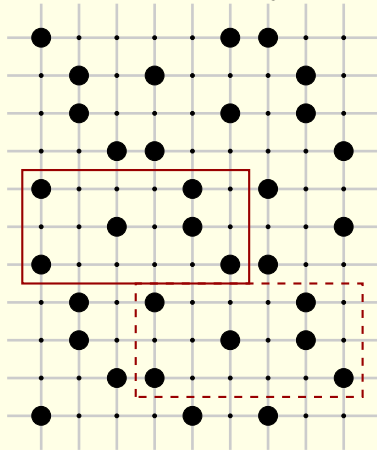
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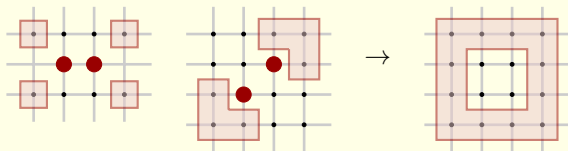
Diagonal pattern  $\mathcal{S}_{r,\Delta}(\bullet, \bullet)$ :



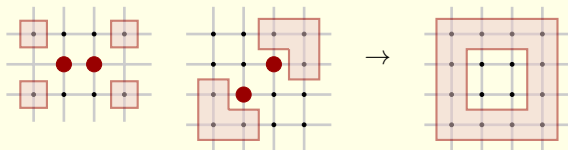
$$\frac{1}{4} \leq D(r, \Delta) \leq \frac{1}{3}$$



## Improvement of the lower bound

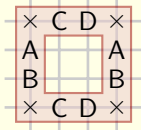


# Improvement of the lower bound

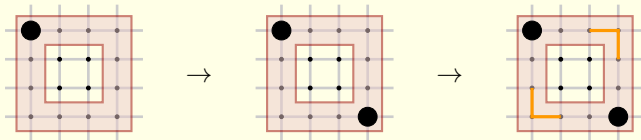


At least 3 code vertices in the frame:

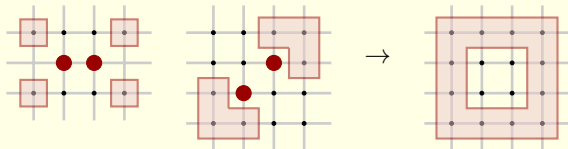
- If no code vertices in the corners, at least four codes in the frame:



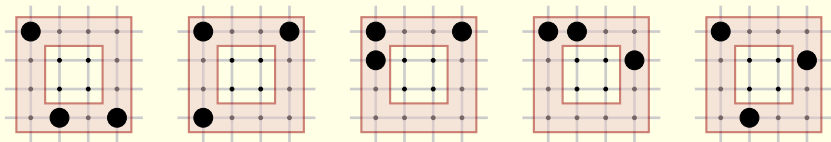
- if one code in a corner:



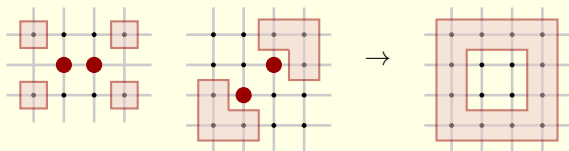
# Improvement of the lower bound



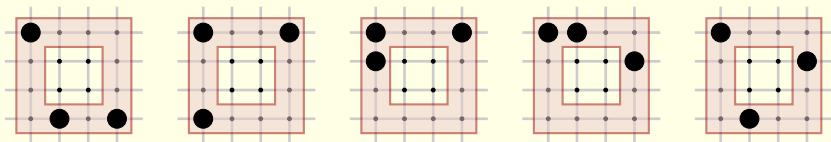
At least 3 codes vertices but only 5 possible configurations:



## Improvement of the lower bound



At least 3 codes vertices but only 5 possible configurations:

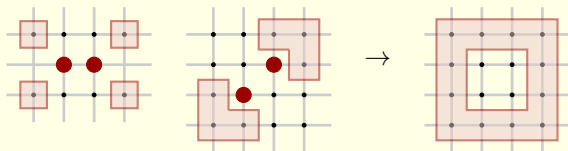


Each frame has a neighbor frame with at least 4 codes.

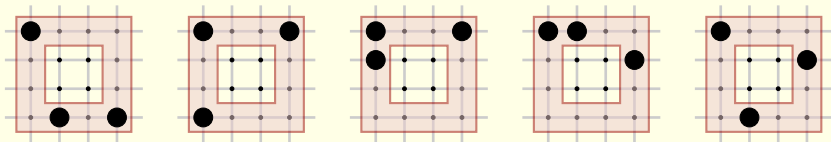
Each frame with at least 4 codes gives charge  $1/5$  to neighbor frames with 3 codes.

$$D(\sqrt{2}, 2 - \sqrt{2}) \geq \frac{(3 + \frac{1}{5})}{12} = \frac{4}{15} \simeq 0.26$$

## Improvement of the lower bound



At least 3 codes vertices but only 5 possible configurations:

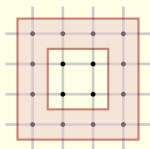


With other rules:

$$D(\sqrt{2}, 2 - \sqrt{2}) \geq \frac{16}{57} \approx 0.28$$



A frame useful for other identification problems!



+

King Grid

+

$(1, \leq 2)$ -identifying code

=

**Theorem** (Laihonen, Foucaud & Parreau, 2011+)

Improvement of the lower bound from  $\frac{5}{12}$  to  $\frac{47}{111}$

(Upper Bound  $\frac{3}{7}$ )

## Results for small values

$r \setminus r + \Delta$	1	$\sqrt{2}$	2	$\sqrt{5}$	$\sqrt{8}$	3	$\sqrt{10}$
1	1	$\frac{1}{2}^b$	X	X	X	X	X
$\sqrt{2}$	—	$\frac{2}{9}^a$	$[\frac{16}{57}^c, \frac{1}{3}]$	X	X	X	X
2	—	—	$[\frac{6}{35}, \frac{5}{29}]^a$	$\frac{1}{2}^b$	$\frac{1}{2}^b$	X	X
$\sqrt{5}$	—	—	—	$\frac{3}{8}^a$	$[\frac{17}{100}^c, \frac{2}{9}]$	$[\frac{1}{4}^b, \frac{1}{3}]$	X
$\sqrt{8}$	—	—	—	—	$\frac{1}{8}^a$	$[\frac{1}{7}^c, \frac{4}{21}]$	$[\frac{1}{4}^b, \frac{3}{8}]$
3	—	—	—	—	—	$[\frac{1}{13.66}^a, \frac{1}{6}]^a$	$[\frac{1}{6}^b, \frac{13}{48}]$

X: No  $(r, \Delta)$ -identifying code

a: Previous results

b: Trivial lower bound with horizontal pattern

c: Lower bound with discharging rules

## Remarks about $\Delta$

To have existence:  $\mathcal{S}_{r,\Delta}(\bullet \bullet) \neq \emptyset$

$$\Delta_m(r) = \sup\{\Delta | \mathcal{S}_{r,\Delta}(\bullet \bullet) \neq \emptyset\}$$

We have  $\Delta_m(r) \leq 1$  et  $\Delta_m(r) = 1$  iff  $r \in \mathbf{N}$ , and:

$$\Delta_m(r) \geq 1 - O\left(\frac{1}{\sqrt{r}}\right)$$

## Remarks about $\Delta$

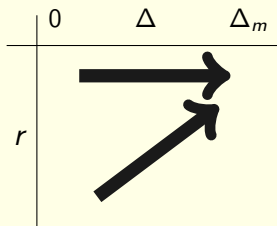
To have existence:  $\mathcal{S}_{r,\Delta}(\bullet, \bullet) \neq \emptyset$

$$\Delta_m(r) = \sup\{\Delta \mid \mathcal{S}_{r,\Delta}(\bullet, \bullet) \neq \emptyset\}$$

We have  $\Delta_m(r) \leq 1$  et  $\Delta_m(r) = 1$  iff  $r \in \mathbf{N}$ , and:

$$\Delta_m(r) \geq 1 - O\left(\frac{1}{\sqrt{r}}\right)$$

Monotonicity of  $D(r, \Delta)$ :



## General values of $r$

**Lower bound:**  $D(r, \Delta) \geq \frac{1}{|S_{r,\Delta}(\bullet, \bullet)|} \rightarrow$  upper bound  $|S_{r,\Delta}(\bullet, \bullet)|$

For big values of  $\Delta$  :  $D(r, \Delta) \geq \theta\left(\frac{1}{\sqrt{r}}\right)$

For other values:  $D(r, \Delta) \geq D(r, 0) \geq \frac{1}{3.22r+4}$

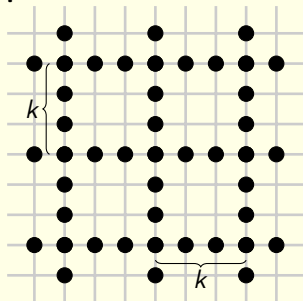
# General values of $r$

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For other values:  $D(r, \Delta) \geq D(r, 0) \geq \frac{1}{3.22r+4}$

**Upper bound:**



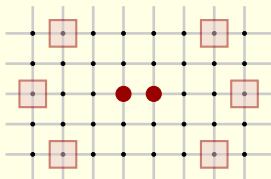
$$D(r, \Delta) \leq \frac{2}{k}$$

$$D(r, \Delta) \leq \frac{4}{r(2 - \Delta - \sqrt{2 - \Delta^2}) + C}$$

# We can do better when we know the patterns

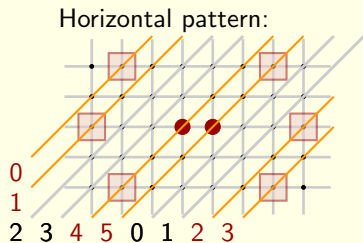
Example:  $(r, \Delta) = (3, \sqrt{10} - 3)$ :

Horizontal pattern:



# We can do better when we know the patterns

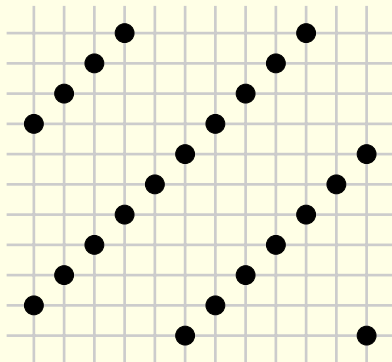
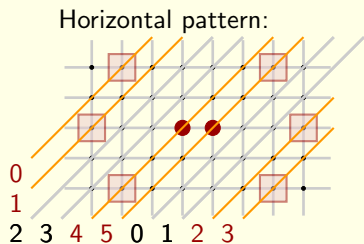
Example:  $(r, \Delta) = (3, \sqrt{10} - 3)$ :





# We can do better when we know the patterns

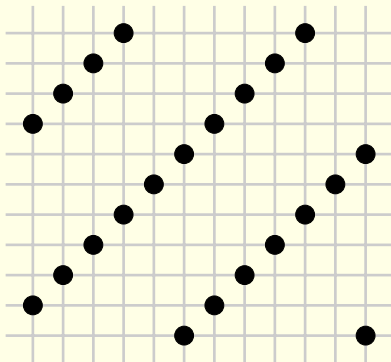
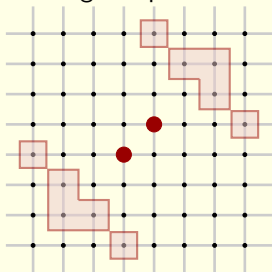
Example:  $(r, \Delta) = (3, \sqrt{10} - 3)$ :



# We can do better when we know the patterns

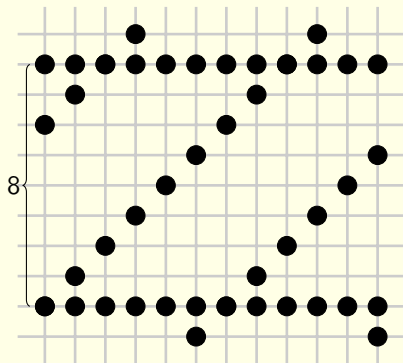
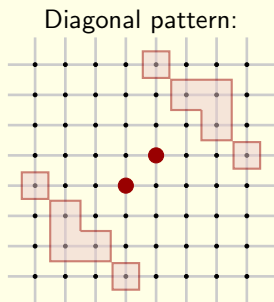
Example:  $(r, \Delta) = (3, \sqrt{10} - 3)$ :

Diagonal pattern:



# We can do better when we know the patterns

Example:  $(r, \Delta) = (3, \sqrt{10} - 3)$ :



Code of density  $\frac{1}{6} + \frac{1}{8} - \frac{1}{48} = \frac{13}{48}$  (we can do  $\frac{1}{4}$ )

## And even better

Sometimes, we can remove horizontal lines:

- $r = \sqrt{101}$ ,  $r + \Delta = 11$ , optimal density  $\frac{1}{4}$
- $r = \sqrt{257}$ ,  $r + \Delta = \sqrt{281}$ , optimal density  $\frac{1}{8}$

Why ?

- exactly  $d$  vertices in  $\mathcal{S}_{r,\Delta}(\bullet \bullet)$  and all diagonals modulo  $d$ .
- $d$  consecutives diagonals and anti-diagonals in  $\mathcal{S}_{r,\Delta}(\bullet \bullet)$

⇒ Construction of infinite family with optimal codes

*Example :  $r = 2k^2 + 1, r + \Delta = \sqrt{r^2 + 2r - 3}$  with  $k \not\equiv 0[3]$ ,  $k$  odd: for  $r$  enough big, optimal code of density  $\frac{1}{6}$*

That's all

Thanks !